Chapter 6 Applications of Trigonometry

Section 6.1 Vectors in the Plane

Exploration 1

1. Use the HMT rule, which states that if an arrow has initial point \((x_1, y_1)\) and terminal point \((x_2, y_2)\), it represents the vector \((x_2 - x_1, y_2 - y_1)\). If the initial point is \((2, 3)\) and the terminal point is \((7, 5)\), the vector is \((7 - 2, 5 - 3) = (5, 2)\).

2. Use the HMT rule, which states that if an arrow has initial point \((x_1, y_1)\) and terminal point \((x_2, y_2)\), it represents the vector \((x_2 - x_1, y_2 - y_1)\). If the initial point is \((3, 5)\) and the terminal point is \((x_2, y_2)\), the vector is \((x_2 - 3, y_2 - 5)\). Using the given vector \((-3, 6)\), we have \(x_2 - 3 = -3\) and \(y_2 - 5 = 6\).

3. Use the HMT rule, which states that if an arrow has initial point \((x_1, y_1)\) and terminal point \((x_2, y_2)\), it represents the vector \((x_2 - x_1, y_2 - y_1)\). If the initial point \(P\) is \((4, -3)\) and the terminal point \(Q\) is \((x_2, y_2)\), the vector \(PQ\) is \((x_2 - 4, y_2 - (-3))\). Using the given vector \(PQ\) \((2, -4)\), we have \(x_2 - 4 = 2\) and \(y_2 + 3 = -4\) \(\Rightarrow y_2 = -7\).

4. Use the HMT rule, which states that if an arrow has initial point \((x_1, y_1)\) and terminal point \((x_2, y_2)\), it represents the vector \((x_2 - x_1, y_2 - y_1)\). If the initial point \(P\) is \((x_1, y_1)\) and the terminal point \(Q\) is \((4, -3)\), the vector \(PQ\) is \((4 - x_1, -3 - y_1)\). Using the given vector \(PQ\) \((2, -4)\), we have \(4 - x_1 = 2\) and \(-3 - y_1 = -4\) \(\Rightarrow y_1 = 1\).

The point \(P\) is \((2, 1)\).

Quick Review 6.1

1. \(x = 9 \cos 30^\circ = \frac{9\sqrt{3}}{2}\), \(y = 9 \sin 30^\circ = 4.5\)

2. \(x = 15 \cos 120^\circ = -7.5\), \(y = 15 \sin 120^\circ = \frac{15\sqrt{3}}{2}\)

3. \(x = 7 \cos 220^\circ \approx -5.36\), \(y = 7 \sin 220^\circ \approx -4.5\)

4. \(x = 6 \cos(-50^\circ) \approx 3.86\), \(y = 6 \sin(-50^\circ) \approx -4.60\)

For #5–6, use a calculator.

5. \(\theta \approx 33.85^\circ\)

6. \(\theta \approx 104.96^\circ\)

For #7–10, the angle determined by \(P(x, y)\) involves \(\tan^{-1}(y/x)\). Since this will always be between \(-180^\circ\) and \(+180^\circ\), you may need to add \(180^\circ\) or \(360^\circ\) to put the angle in the correct quadrant.

7. \(\theta = \tan^{-1}\left(\frac{9}{5}\right) \approx 60.95^\circ\)

8. \(\theta = 360^\circ + \tan^{-1}\left(-\frac{7}{5}\right) \approx 305.54^\circ\)

9. \(\theta = 180^\circ + \tan^{-1}\left(\frac{5}{7}\right) \approx 248.20^\circ\)

10. After 3 hours, the ship has traveled \((3)(42 \sin 40^\circ)\) nautical miles north. Five hours later, it is \((3)(42 \sin 40^\circ) + (5)(42 \sin 125^\circ) \approx 253.013\) nautical miles north and \((3)(42 \cos 40^\circ) + (5)(42 \cos 125^\circ) \approx -23.929\) nautical miles north (about 23.93 nautical miles south) of Port Norfolk.

Bearing: \(180^\circ + \tan^{-1}\left(-\frac{253.013}{23.929}\right) \approx 95.40^\circ\)

Distance: \(\sqrt{(253.013)^2 + (-23.929)^2} \approx 254.14\) nautical miles.

Section 6.1 Exercises

For #1–4, recall that two vectors are equivalent if they have the same magnitude and direction. If \(R\) has coordinates \((a, b)\) and \(S\) has coordinates \((c, d)\), then the magnitude of \(RS\) is \(|RS| = \sqrt{(c - a)^2 + (d - b)^2} = RS\), the distance from \(R\) to \(S\). The direction of \(RS\) is determined by the coordinates \((c - a, d - b)\).

1. If \(R = (-4, 7)\) and \(S = (-1, 5)\), then, using the HMT rule, \(RS = (-1 - (-4), 5 - 7) = (3, -2)\).

2. If \(P = (0, 0)\) and \(Q = (3, -2)\), then, using the HMT rule, \(PQ = (3 - 0, -2 - 0) = (3, -2)\).

Both vectors represent \((3, -2)\) by the HMT rule.

2. If \(R = (7, -3)\) and \(S = (4, -5)\), then, using the HMT rule, \(RS = (4 - 7, -5 - (-3)) = (-3, -2)\).

If \(P = (0, 0)\) and \(Q = (-3, -2)\), then, using the HMT rule, \(PQ = (-3 - 0, -2 - 0) = (-3, -2)\).

Both vectors represent \((-3, -2)\) by the HMT rule.

3. If \(R = (2, 1)\) and \(S = (0, -1)\), then, using the HMT rule, \(RS = (0 - 2, -1 - 1) = (-2, -2)\).

If \(P = (1, 4)\) and \(Q = (-1, 2)\), then, using the HMT rule, \(PQ = (-1 - 1, 2 - 4) = (-2, -2)\).

Both vectors represent \((-2, -2)\) by the HMT rule.

4. If \(R = (-2, -1)\) and \(S = (2, 4)\), then, using the HMT rule, \(RS = (2 - (-2), 4 - (-1)) = (4, 5)\).

If \(P = (-3, -1)\) and \(Q = (1, 4)\), then, using the HMT rule, \(PQ = (1 - (-3), 4 - (-1)) = (4, 5)\).

Both vectors represent \((4, 5)\) by the HMT rule.

5. \(PQ = (3 - (-2), 4 - 2) = (5, 2)\).

\(|PQ| = \sqrt{5^2 + 2^2} = \sqrt{29}\)
6. \( \overrightarrow{RS} = (2 - (-2), -8 - 5) = (4, -13), \)
   \[ |\overrightarrow{RS}| = \sqrt{4^2 + (-13)^2} = \sqrt{185} \]
7. \( \overrightarrow{QR} = (-2 - 3, 5 - 4) = (-5, 1), \)
   \[ |\overrightarrow{QR}| = \sqrt{(-5)^2 + 1^2} = \sqrt{26} \]
8. \( \overrightarrow{PS} = (2 - (-2), -8 - 2) = (4, -10), \)
   \[ |\overrightarrow{PS}| = \sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29} \]
9. \( 2 \overrightarrow{QS} = 2(2 - 3, -8 - 4) = (-2, -24), \)
   \[ |2 \overrightarrow{QS}| = \sqrt{(-2)^2 + (-24)^2} = \sqrt{580} = 2\sqrt{145} \]
10. \( (\sqrt{2}) \overrightarrow{PR} = \sqrt{2} (2 - (-2), -8 - 2) = (0, 3\sqrt{2}), \)
    \[ |\sqrt{2} \overrightarrow{PR}| = \sqrt{0^2 + (3\sqrt{2})^2} = 3\sqrt{2} \]
11. \( 3 \overrightarrow{QR} + \overrightarrow{PS} = 3(-5, 1) + (4, -10) = (-11, -7), \)
    \[ |3 \overrightarrow{QR} + \overrightarrow{PS}| = \sqrt{(-11)^2 + (-7)^2} = \sqrt{170} \]
12. \( \overrightarrow{PS} - 3 \overrightarrow{PQ} = (4, -10) - 3(5, 2) = (-11, -16), \)
    \[ |\overrightarrow{PS} - 3 \overrightarrow{PQ}| = \sqrt{(-11)^2 + (-16)^2} = \sqrt{377} \]
13. \((-1, 3) + (2, 4) = (1, 7) \]
14. \((-1, 3) - (2, 4) = (-3, -1) \]
15. \((-1, 3) - (2, -5) = (-3, 8) \]
16. \(3(2, 4) = (6, 12) \]
17. \(2(-1, 3) + 3(2, -5) = (4, -9) \]
18. \(2(-1, 3) - 4(2, 4) = (-10, -10) \]
19. \(-2(-1, 3) - 3(2, 4) = (-4, -18) \]
20. \((-1, -3) - (2, -4) = (-1, -7) \]
21. \( u = \frac{2}{\sqrt{(-2)^2 + 4^2}} \frac{4}{\sqrt{(-2)^2 + 4^2}} \approx 0.451 + 0.89j \]
22. \( v = \frac{1}{\sqrt{1^2 + (-1)^2}} \frac{-1}{\sqrt{1^2 + (-1)^2}} \approx 0.71i - 0.71j \]
23. \( w = \frac{1}{\sqrt{(-1)^2 + (-2)^2}} \frac{-2}{\sqrt{(-1)^2 + (-2)^2}} \approx -0.451 - 0.89j \]
24. \( w = \frac{5}{\sqrt{3^2 + 5^2}} \frac{5}{\sqrt{3^2 + 5^2}} \approx 0.71i + 0.71j \]

For #25–28, the unit vector in the direction of \( v = (a, b) \) is
\[ v = \frac{1}{|v|} \left( \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right) = \frac{a}{\sqrt{a^2 + b^2}} i + \frac{b}{\sqrt{a^2 + b^2}} j. \]

25. \( \left( \frac{2}{\sqrt{3}} \right) \)
   \[ \frac{2}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j \]
26. \( \left( \frac{-3}{\sqrt{13}} \right) \)
   \[ \frac{-3}{\sqrt{13}} i + \frac{2}{\sqrt{13}} j \]

27. (a) \( \left( -\frac{4}{\sqrt{41}}, -\frac{5}{\sqrt{41}} \right) \)
   (b) \( -\frac{4}{\sqrt{41}} i + \left( -\frac{5}{\sqrt{41}} \right) j = -\frac{4}{\sqrt{41}} i - \frac{5}{\sqrt{41}} j \)
28. (a) \( \left( \frac{3}{5}, -\frac{4}{5} \right) \)
   (b) \( \frac{3}{5} i + \left( -\frac{4}{5} \right) j = \frac{3}{5} i - \frac{4}{5} j \)

29. \( v = (18 \cos 25°, 18 \sin 25°) \approx (16.31, 7.61) \)
30. \( v = (14 \cos 55°, 14 \sin 55°) \approx (8.03, 11.47) \)
31. \( v = (47 \cos 108°, 47 \sin 108°) \approx (-14.52, 44.70) \)
32. \( v = (33 \cos 136°, 33 \sin 136°) \approx (-23.74, 22.92) \)
33. \( |u| = 3\sqrt{2^2 + 4^2} = 5, a = \cos^{-1} \left( \frac{3}{5} \right) \approx 53.13° \)
34. \( |u| = 5, a = 360° - \cos^{-1} \left( \frac{3}{5} \right) \approx 306.87° \)
35. \( |u| = 3\sqrt{2^2 + (-4)^2} = 5, \alpha = 360° - \cos^{-1} \left( \frac{3}{5} \right) \approx 306.87° \)
36. \( \alpha = 360° - \cos^{-1} \left( \frac{-3}{\sqrt{34}} \right) \approx 239.04° \)
37. \( \alpha = \left( 7 \cos 135° \right) i + \left( 7 \sin 135° \right) j = \left( 3 \cos \theta \right) i + \left( 3 \sin \theta \right) j, |u| = 7 \) and \( \alpha = 135°. \)
38. \( \alpha = \left( 2 \cos 60° \right) i + \left( 2 \sin 60° \right) j = \left( 1 \cos \theta \right) i + \left( 1 \sin \theta \right) j, |u| = 2 \) and \( \alpha = 60°. \)

For #39–40, first find the unit vector in the direction of \( u. \) Then multiply by the magnitude of \( v, |v|. \)

39. \( v = |v| \cdot u = 2 \sqrt{\frac{3}{3^2 + (-3)^2}} \cdot \sqrt{\frac{-3}{3^2 + (-3)^2}} \approx 2 \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) = (\sqrt{2}, -\sqrt{2}) \)
40. \( v = |v| \cdot u = 5 \sqrt{\frac{-5}{(-5)^2 + 7^2}} \cdot \sqrt{\frac{7}{(-5)^2 + 7^2}} \approx 5(-0.58, 0.81) = (-2.91, 4.07) \)

41. A bearing of 335° corresponds to a direction angle of 115°. \( v = 530 (\cos 115°, \sin 115°) \approx (-223.99, 480.34) \)
42. A bearing of 170° corresponds to a direction angle of 80°. \( v = 460 (\cos 80°, \sin 80°) \approx (79.88, -453.01) \)
43. (a) A bearing of 340° corresponds to a direction angle of 110°. \( v = 325 (\cos 110°, \sin 110°) \approx (-111.16, 305.40) \)
   (b) The wind bearing of 320° corresponds to a direction angle of 130°. The wind vector is
   \( w = 40 (\cos 130°, \sin 130°) \approx (-25.71, 30.64) \). Actual velocity vector: \( v + w \approx (-136.87, 336.04) \).
   Actual speed: \( |v + w| \approx \sqrt{136.87^2 + 336.04^2} \approx 362.84 \) mph.
   Actual direction: \( \theta = 180° + \tan^{-1} \left( \frac{336.04}{-136.87} \right) \approx 112.16° \), so the bearing is about 337.84°.
44. (a) A bearing of 170° corresponds to a direction angle of -80°.
\[ v = 460 \cos(-80°), \sin(-80°) \approx (79.88, -453.01). \]
(b) The wind bearing of 200° corresponds to a direction angle of -110°. The wind vector is \( w = 80 \cos(-110°), \sin(-110°) \approx (-27.36, -75.18). \) Actual velocity vector: \( v + w \approx (52.52, -528.19). \) Actual speed: \( |v + w| \approx \sqrt{52.52^2 + 528.19^2} \approx 530.79 \text{ mph}. \) Actual direction: \( \theta = 180° + \tan^{-1}\left(\frac{52.52}{528.19}\right) \approx -84.32°, \) so the bearing is about 174.32°.

45. (a) \( v = 10 \cos(70°), \sin(70°) \approx (3.42, 9.40). \)
(b) The horizontal component is the (constant) horizontal speed of the basketball as it travels toward the basket. The vertical component is the vertical velocity of the basketball, affected by both the initial speed and the downward pull of gravity.

46. (a) \( v = 2.5 \cos(15°), \sin(15°) \approx (2.41, 0.65). \)
(b) The horizontal component is the force moving the box forward. The vertical component is the force moving the box upward against the pull of gravity.

47. We need to choose \( w = (a, b) = k (\cos(33°), \sin(33°)), \) so that \( k \cos(33° - 15°) = k \cos(18°) = 2.5. \) (Redefine “horizontal” to mean the parallel to the inclined plane; then the moving vector makes an angle of 18° with the “horizontal.”) Then \( k = \frac{2.5}{\cos(18°)} \approx 2.63 \text{ lb}, \) so that \( w \approx (2.20, 1.43). \)

48. Juana’s force can be represented by \( 23 \cos(18°), \sin(18°) \approx (21.87, 7.11), \) while Diego’s force is \( 27 \cos(-15°), \sin(-15°) \approx (26.08, -6.99). \) Their total force is therefore \( (47.95, 0.12), \) so Corral must be pulling with an equal force in the opposite direction: \((-47.95, -0.12). \) The magnitude of Corral’s force is about 47.95 lb.

49. \( F = (50 \cos(45°), 50 \sin(45°)) + (75 \cos(30°), 75 \sin(30°)) \approx (100.31, -2.14), \) so \( |F| \approx 100.33 \text{ lb} \) and \( \theta \approx -1.22°. \)

50. \( F = 100 \cos(50°), \sin(50°) + 50 \cos(60°), \sin(60°) + 80 \cos(-20°), \sin(-20°) \approx (92.47, 66.34), \) so \( |F| \approx 113.81 \text{ lb} \) and \( \theta \approx 35.66°. \)

51. Ship heading: \( 12 \cos(90°), 12 \sin(90°) = (0, 12) \)

   Current heading: \( 4 \cos(225°), 4 \sin(225°) \approx (-2.83, -2.83) \)

   The ship’s actual velocity vector is \( (-2.83, 9.17), \) so its speed is \( \sqrt{(-2.83)^2 + 9.17^2} \approx 9.6 \text{ mph} \) and the direction angle is \( \cos^{-1}\left(\frac{-2.83}{9.6}\right) \approx 107.14°, \) so the bearing is about 342.86°.

52. Let \( v = (0, 20) \) be the velocity of the boat and \( w = (8, 0) \) be the velocity vector of the current. If the boat travels \( t \) minutes to reach the opposite shore, then its position, in vector form, must be \( (0, 20t) + (8t, 0) = (8t, 20t) = (8t, 1). \)

So \( 20t = 1 \Rightarrow t = \frac{1}{20} \Rightarrow 8t = 0.4 \text{ mi}. \) The boat meets the shore 0.4 mi downstream.

53. Let \( w \) be the speed of the ship. The ship’s velocity (in still water) is \( (w \cos(270°), w \sin(270°)) = (0, -w). \)

Let \( z \) be the speed of the current. Then, the current velocity is \( (z \cos(135°), z \sin(135°)) \approx (-0.712, 0.71z). \)

The position of the ship after two hours is \( (20 \cos(240°), 20 \sin(240°)) \approx (-10, -17.32). \)

Putting all this together we have:

\[
\begin{align*}
(2,0,-w) + & 2(-0.712, 0.71z) = (-10, -17.32), \\
(-1.42, -2w + 1.42z) = (-10, -17.32), \text{ so } z \approx 7.07 \text{ and } w \approx 13.66. \text{ The speed of the ship is about } 13.66 \text{ mph, and the speed of the current is about } 7.07. \n\end{align*}
\]

54. Let \( u = (u_1, u_2), v = (v_1, v_2), \) and \( w = (w_1, w_2). \)

   (a) \( u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) = (u_1 + v_1, u_2 + v_2) = v + u \)

   (b) \( u + w = (u_1, u_2) + (w_1, w_2) = (u_1 + w_1, u_2 + w_2) = (u_1, u_2) + (v_1, v_2) + (w_1, w_2) = u + (v + w) \)

   (c) \( u + 0 = (u_1, u_2) + (0, 0) = (u_1, u_2) = (u_1, u_2) = u \)

   (d) \( u + ( -u) = (u_1, u_2) + ( -u_1, -u_2) = u_1 + (-u_1), u_2 + (-u_2) = (0, 0) = 0 \)

   (e) \( a(u + v) = a(u_1 + v_1, u_2 + v_2) = a(u_1 + v_1, u_2 + v_2) = \langle a(u_1, u_2) + a(v_1, v_2) \rangle = \langle a(u_1 + v_1), a(u_2 + v_2) \rangle = \langle a(u_1, u_2), a(v_1, v_2) \rangle = a(u_1, u_2) + a(v_1, v_2) = au + av \)

   (f) \( (a + b)u = (a + b)(u_1, u_2) = (au_1 + bu_1, au_2 + bu_2) = (au_1, au_2) + (bu_1, bu_2) = a(u_1, u_2) + b(u_1, u_2) = au + bu \)

   (g) \( (ab)u = a(bu) = (a(bu_1, bu_2)) = (a, bu_2) = a(bu) = ab \)

   (h) \( ab = a(0,0) = (a,0)a(0,0) = (0,0) = 0 \)

   (i) \( (1)u = ((1)u_1, (1)u_2) = (u_1, u_2) = u \)

   (j) \( |u| = |(a(u_1, u_2))| = \sqrt{(au_1)^2 + (au_2)^2} = |a| \sqrt{u_1^2 + u_2^2} = |a|u \)

55. True. Vectors \( u \) and \(-u\) have the same length but opposite directions. Thus, the length of \(-u\) is also 1.

56. False. \(1/u\) is not a vector at all.

57. \(|(2,-1)| = \sqrt{2^2 + (-1)^2} = \sqrt{5}\)

The answer is D.

58. \( u - v = (2,3) - (4,-1) = (-2, 4, 3 - (-1)) = (-6, 4) \)

The answer is E.

59. The x-component is 3 cos 30°, and the y-component is 3 sin 30°. The answer is A.

60. \(|v| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}, \) so the unit vector is \(-1,3/\sqrt{10}. \) The answer is C.
61. (a) Let \( A \) be the point \((a_1, a_2)\) and \( B \) be the point \((b_1, b_2)\). Then \( \overrightarrow{OA} \) is the vector \( \langle a_1, a_2 \rangle \) and \( \overrightarrow{OB} \) is the vector \( \langle b_1, b_2 \rangle \).

So, \( \overrightarrow{BA} = \langle a_1 - b_1, a_2 - b_2 \rangle \)
\[ = \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle \]
\[ = \overrightarrow{OA} - \overrightarrow{OB} \]

(b) \( x \overrightarrow{OA} + y \overrightarrow{OB} = x(\overrightarrow{OC} + \overrightarrow{CA}) + y(\overrightarrow{OC} + \overrightarrow{CB}) \)
(from part (a))
\[ = x \overrightarrow{OC} + x \overrightarrow{CA} + y \overrightarrow{OC} + y \overrightarrow{CB} \]
\[ = (x + y) \overrightarrow{OC} + x \overrightarrow{CA} + y \overrightarrow{CB} \]
\[ = \overrightarrow{OC} + x \overrightarrow{CA} + y \overrightarrow{CB} \text{ (since } x + y = 1) \]
\[ = \overrightarrow{OC} + y \left( \frac{\overrightarrow{BC}}{|CA|} \cdot \overrightarrow{CA} + \overrightarrow{CB} \right) \quad \text{ (since } x = y \left( \frac{|BC|}{|CA|} \right) \}
\[ = \overrightarrow{OC} + y \left( \frac{BC}{|CA|} \cdot \overrightarrow{CA} + \overrightarrow{CB} \right) \]

\(\frac{\overrightarrow{CA}}{|CA|}\) is a unit vector, and \(\overrightarrow{BC}\) points in the same direction.
\[ = \overrightarrow{OC} + y \left( \frac{BC}{|CA|} \cdot \overrightarrow{CA} + \overrightarrow{CB} \right) \]
\[ = \overrightarrow{OC} + y \left( \frac{BC}{|CA|} \cdot \overrightarrow{CA} + \frac{BC}{|CA|} \cdot \overrightarrow{CB} \right) \]
\[ = \overrightarrow{OC} + y \left( \frac{BC}{|CA|} \cdot \overrightarrow{CA} + \frac{BC}{|CA|} \cdot \overrightarrow{CB} \right) \]

62. (a) By Exercise 61, \( OM_1 = x \overrightarrow{OA} + y \overrightarrow{OB} \), where \( x + y = 1 \). Since \( M_1 \) is the midpoint, \( |BM_1| = |M_1A| \).
We know from Exercise 61, however that
\[ |BM_1| = 1 = \frac{x}{y} \] So \( x = y = \frac{1}{2} \). As a result,
\[ OM_1 = \frac{1}{2} \overrightarrow{OA} + \frac{1}{2} \overrightarrow{OB} \]. The proof for \( OM_2 \)
and \( OM_3 \) are similar.
(b) \( 2 \overrightarrow{OM}_1 + \overrightarrow{OC} = 2 \left( \frac{1}{2} \overrightarrow{OA} + \frac{1}{2} \overrightarrow{OB} \right) + \overrightarrow{OC} \)
\[ = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \]
Use the same method for the other proofs.
(c) Part (b) implies that \( 2 \overrightarrow{OM}_1 + \overrightarrow{OC} = 2 \overrightarrow{OM}_2 + \overrightarrow{OA} \)
\[ = 2 \overrightarrow{OM}_3 + \overrightarrow{OB} \]. Each of the three vectors lies along a different median (that is, if nonzero, the three vectors have three different directions).
Hence they can only be equal if all are equal to \( \mathbf{0} \).
Thus \( 2 \overrightarrow{OM}_1 = -\overrightarrow{OC}, 2 \overrightarrow{OM}_2 = -\overrightarrow{OA}, \) and
\[ 2 \overrightarrow{OM}_3 = -\overrightarrow{OB}, \] so
\[ \frac{|OM_1|}{|OC|} = \frac{|OM_2|}{|OA|} = \frac{|OM_3|}{|OB|} = \frac{1}{2} \]

63. Use the result of Exercise 61. First we show that if \( C \) is on the line segment \( AB \), then there is a real number \( t \) so
\[ \frac{BC}{|CA|} = \frac{t}{1 - t} \] (Convince yourself that
\[ \frac{BC}{BC + CA} \] works.) Then \( \overrightarrow{OC} = t \overrightarrow{OA} + (1 - t) \overrightarrow{OB} \).
A similar argument can be used in the cases where \( B \) is on the line segment \( AC \) or \( A \) is on the line segment \( BC \).

Suppose there is a real number \( t \) so that
\[ \overrightarrow{OC} = t \overrightarrow{OA} + (1 - t) \overrightarrow{OB} \]. We also know
\[ \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} \text{ and } \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} \]. So we have
\[ t \overrightarrow{OA} + (1 - t) \overrightarrow{OB} = \overrightarrow{OB} + \overrightarrow{BC} \]
\[ t \overrightarrow{OA} + (1 - t) \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} \]. Therefore,
\[ t (\overrightarrow{OA} - \overrightarrow{OB}) = \overrightarrow{BC} \text{ and } (t - 1)(\overrightarrow{OA} - \overrightarrow{OB}) = \overrightarrow{AC} \].
Hence \( BC \) and \( AC \) have the same or opposite directions, so \( C \) must lie on the line \( L \) through the two points \( A \) and \( B \).

![Diagram](https://via.placeholder.com/150)
Section 6.2 Exercises

1. 60 + 12 = 72
2. −40 + 26 = −14
3. −12 − 35 = −47
4. 10 − 56 = −46
5. 12 + 18 = 30
6. −16 − 28 = −44
7. −14 + 0 = −14
8. 0 + 33 = 33
9. |u| = √u·u = √25 + 144 = 13
10. |u| = √v·v = √64 + 225 = 17
11. |u| = √u·u = √16 = 4
12. |u| = √v·v = √9 = 3
13. u·v = 4 − 15 = −11, |u| = √16 + 9 = 5,
   |v| = √1 + 25 = √26, θ = cos⁻¹(−11/5√26) ≈ 115.6°
14. u·v = −6 + 6 = 0, θ = 90°
15. u·v = −6 + 15 = 9,
   |u| = √4 + 9 = √13,
   |v| = √9 + 25 = 34,
   θ = cos⁻¹(9/√13·34) ≈ 64.65°
16. u·v = −30 + 2 = −32, |u| = √25 + 4 = √29,
   |v| = √36 + 1 = √37,
   θ = cos⁻¹(−32/√29·√37) ≈ 167.66°
17. u·v = −6 ÷ 6 = 1, |u| = √9 + 9 = √18,
   |v| = √4 + 12 = √16 = 4,
   θ = cos⁻¹(−6 ÷ 6/√18·4) = 165°
18. u·v = 0, θ = 90°
19. u has direction angle π/4 and v has direction angle 3π/2
   (which is equivalent to −π/2), so the angle between the
   vectors is π/4 − (−π/2) = 3π/4 or 135°.
20. u has direction angle π/3 and v has direction angle 5π/6,
   so the angle between the vectors is 5π/6 − π/3 = π/2 or 90°.
21. u·v = −24 + 20 = −4, |u| = √64 + 25 = √89,
   |v| = √9 + 16 = 5,
   θ = cos⁻¹(−4/5√89) ≈ 94.86°
22. u·v = 3 − 72 = −69, |u| = √9 + 64 = √73,
   |v| = √1 + 81 = 82,
   θ = cos⁻¹(−69/√73·√82) ≈ 153.10°
23. u·v = (2, 3) · (3/2, −1) = 2(3/2) + 3(−1)
    = 3 − 3 = 0
   Since u·v = 0, u and v are orthogonal.

24. u·v = (−4, −1) · (1, −4) = −4(1) + (−1)(−4)
   = −4 + 4 = 0
   Since u·v = 0, u and v are orthogonal.

For #25–28, first find proj_u. Then use the fact that u·v = 0 when u and v are orthogonal.
25. proj_u = (−8, 3) · (−6, −2)
   = (42/40)(−6, −2) = (21/20)(−6, −2)
   = −21/10 (3, 1)
   u = −21/10 (3, 1) + 17/10 (−1, 3)
26. proj_u = (3, −7) · (−2, −6)
   = (36/40)(−2, −6) = (9/5)(1, 3)
   u = −9/5 (1, 3) + 8/5 (3, −1)
27. proj_u = (8, 5) · (−9, −2)
   = (8, 5)(−9, −2) = (82/85)(−9, −2)
   u = 82/85 (9, 2) + 29/85 (−2, 9)
28. proj_u = (−2, 8) · (9, −3)
   = (−2, 8)(9, −3) = (−42/90)(9, −3) = 7/5 (−3, 1)
   u = 7/5 (−3, 1) + 1/5 (11, 33)

29.

\[ \overrightarrow{CA} \cdot \overrightarrow{CB} = (1 − 3, 10 − 1) \cdot (−4 − 3, 5 − 1) \]
\[ = (−2, 9) \cdot (−7, 4) = 14 + 36 = 50, \]
\[ |\overrightarrow{CA}| = √4 + 81 = √85, |\overrightarrow{CB}| = √49 + 16 = √65, \]
\[ C = \cos⁻¹(0/√85·√65) \approx 47.73° \]
\[ \overrightarrow{BC} \cdot \overrightarrow{BA} = (7, −4) \cdot (1 − (−4), 10 − 5) \]
\[ = (7, −4) \cdot (5, 5) = 35 − 20 = 15, \]
\[ |\overrightarrow{BC}| = √65, |\overrightarrow{BA}| = √50, B = \cos⁻¹(0/\sqrt{65·\sqrt{50}}) \approx 74.74° \]
\[ A = 180° − B − C \approx 180° − 74.74° − 47.73° = 57.53° \]
30. \( \overrightarrow{CA} \cdot \overrightarrow{CB} = (-4 - 1, 1 - (-6)) \cdot (5 - 1, -1 - (-6)) = (-5, 7) \cdot (4, 5) = -20 + 35 = 15, \]
\( |\overrightarrow{CA}| = \sqrt{25 + 49} = \sqrt{74}, |\overrightarrow{CB}| = \sqrt{16 + 25} = \sqrt{41}, \)
\( C = \cos^{-1} \left( \frac{15}{\sqrt{74} \cdot \sqrt{41}} \right) \approx 74.20^\circ \)
\( \overrightarrow{BA} \cdot \overrightarrow{BC} = (-4 - 5, 1 - (-1)) \cdot (1 - 5, -6 - (-1)) = (-9, 2) \cdot (-4, -5) = 36 - 10 = 26, \)
\( |\overrightarrow{BA}| = \sqrt{81 + 4} = \sqrt{85}, |\overrightarrow{BC}| = \sqrt{41}, \)
\( B = \cos^{-1} \left( \frac{26}{\sqrt{85} \cdot \sqrt{41}} \right) \approx 63.87^\circ \)
\( A = 180^\circ - B - C \approx 180^\circ - 63.87^\circ - 74.20^\circ = 41.93^\circ \)

For #31–32, use the relationship \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta. \)
31. \( \mathbf{u} \cdot \mathbf{v} = 3 \cdot 8 \cos 150^\circ \approx -20.78 \)
32. \( \mathbf{u} \cdot \mathbf{v} = 12 \cdot 40 \cos \left( \frac{\pi}{3} \right) = 240 \)

For #33–38, vectors are orthogonal if \( \mathbf{u} \cdot \mathbf{v} = 0 \) and are parallel if \( \mathbf{u} = k \mathbf{v} \) for some constant \( k. \)
33. Parallel: \(-2 \left( \frac{10}{4}, \frac{-3}{2} \right) = \left( \frac{-10}{2}, 3 \right) = (5, 3) \)
34. Neither: \( \mathbf{u} \cdot \mathbf{v} = \frac{40}{3} \neq 0 \) and \( \frac{3}{5} \mathbf{v} = \frac{3}{5} \left( \frac{10}{4}, \frac{-3}{2} \right) = \left( \frac{2}{5}, \frac{-3}{2} \right) \neq \mathbf{u} \)
35. Neither: \( \mathbf{u} \cdot \mathbf{v} = -120 \neq 0 \) and \( \frac{-15}{4} \mathbf{v} = \frac{-15}{4} \left( -4, -5 \right) = \left( 15, \frac{-75}{4} \right) \neq \mathbf{u} \)
36. Orthogonal: \( \mathbf{u} \cdot \mathbf{v} = -60 + 60 = 0 \)
37. Orthogonal: \( \mathbf{u} \cdot \mathbf{v} = -60 + 60 = 0 \)
38. Parallel: \( -\frac{1}{2} \mathbf{v} = -\frac{1}{2} \left( -4, 14 \right) = (2, -7) = \mathbf{u} \)

For #39–42 part (b), first find the direction(s) of \( \overrightarrow{AP} \) and then find the unit vectors. Then find \( P \) by adding the coordinates of \( A \) to the components of a unit vector.
39. (a) \( A \) is \((4, 0)\) and \( B \) is \((0, -3)\).
   (b) The line is parallel to \( \overrightarrow{AB} = (0 - 4, -3 - 0) = (-4, -3) \), so the direction of \( \overrightarrow{AP} \) is \( \mathbf{u} = (3, -4) \) or \( \mathbf{v} = (-3, 4) \).

\[
\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{(3, -4)}{\sqrt{9 + 16}} = \frac{3}{5}, \frac{-4}{5} \]
\[
\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-3, 4)}{\sqrt{9 + 16}} = \frac{3}{5}, \frac{4}{5} \]

So, \( P \) is \((4.6, -0.8)\) or \((3.4, 0.8)\).
40. (a) \( A \) is \((-5, 0)\) and \( B \) is \((0, 2)\).
   (b) The line is parallel to \( \overrightarrow{AB} = (0 - (-5), 2 - 0) = (5, 2) \), so the direction of \( \overrightarrow{AP} \) is \( \mathbf{u} = (-2, 5) \) or \( \mathbf{v} = (2, -5) \).

\[
\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{(-2, 5)}{\sqrt{4 + 25}} = \left( -\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right) \]
\[
\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(2, -5)}{\sqrt{29}} = \left( \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}} \right) \]

So \( P \) is \((-5 - \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}) \approx (-5.37, 0.93) \) or \((-5 + \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}) \approx (-4.63, -0.93) \).
41. (a) \( A \) is \((7, 0)\) and \( B \) is \((0, -3)\).
   (b) The line is parallel to \( \overrightarrow{AB} = (0 - 7, -3 - 0) = (-7, -3) \), so the direction of \( \overrightarrow{AP} \) is \( \mathbf{u} = (3, -7) \) or \( \mathbf{v} = (-3, 7) \).

\[
\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{(3, -7)}{\sqrt{9 + 49}} = \left( \frac{3}{\sqrt{58}}, -\frac{7}{\sqrt{58}} \right) \]
\[
\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-3, 7)}{\sqrt{9 + 49}} = \left( -\frac{3}{\sqrt{58}}, \frac{7}{\sqrt{58}} \right) \]

So \( P \) is \((7 + \frac{3}{\sqrt{58}}, -\frac{7}{\sqrt{58}}) \approx (7.39, 0.92) \) or \((7 - \frac{3}{\sqrt{58}}, \frac{7}{\sqrt{58}}) \approx (6.61, 0.92) \).
42. (a) \( A \) is \((6, 0)\) and \( B \) is \((0, 3)\).
   (b) The line is parallel to \( \overrightarrow{AB} = (0 - 6, 3 - 0) = (-6, 3) \), so the direction of \( \overrightarrow{AP} \) is \( \mathbf{u} = (3, 6) \) or \( \mathbf{v} = (-3, -6) \).

\[
\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{(3, 6)}{\sqrt{9 + 36}} = \left( \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right) \]
\[
\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(-3, -6)}{\sqrt{9 + 36}} = \left( -\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}} \right) \]

So \( P \) is \((6 + \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \approx (6.45, 0.89) \) or \((6 - \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}) \approx (5.55, -0.89) \).
43. \( 2v_1 + 3v_2 = 10, v_1^2 + v_2^2 = 17 \). Since \( v_1 = 5 - \frac{3}{2} v_2 \), \( \left( 5 - \frac{3}{2} v_2 \right)^2 + v_2^2 = 17, 25 - 15v_2 + \frac{9}{4} v_2^2 + v_2^2 = 17, \)
\( \frac{13}{4} v_2^2 - 15v_2 + 8 = 0, 13v_2^2 - 60v_2 + 32 = 0, \)
\( (v_2 - 4)(13v_2 - 8) = 0 \), so \( v_2 = 4 \) or \( v_2 = \frac{8}{13} \).

Therefore, \( v \approx (-1, 4) \) or \( v = \frac{53}{13}, \frac{8}{13} \approx (4.07, 0.62) \).
44. \(-2v_1 + 5v_2 = -11, v_1^2 + v_2^2 = 10\). Since \(v_1 = \frac{5}{2} v_2 + \frac{11}{2}\),
\[
\left(\frac{1}{2}(5v_1 + 11)\right)^2 + v_2^2 = 10,
\]
\[
\frac{25v_1^2}{4} + \frac{110v_2}{4} + \frac{121}{4} + v_2^2 = 10,
\]
\[
\frac{29}{4}v_2^2 + \frac{110}{4}v_2 + \frac{81}{4} = 0, 29v_2^2 + 110v_2 + 81 = 0,
\]
\(v_2 + 1)(29v_2 + 81) = 0, \text{ so } v_2 = -1 \text{ or } v_2 = -\frac{81}{29}.
\]
Therefore, \(v = (3, -1) \text{ or } v = \left(-\frac{43}{29}, -\frac{81}{29}\right) \approx (-1.48, -2.79).

45. \(v = (\cos 60^\circ)i + (\sin 60^\circ)j\) = \(\frac{1}{2}i + \frac{\sqrt{3}}{2}j\).
\[
F_1 = \text{proj}_vF = (F \cdot v)v = \left(-160j \cdot \left(\frac{1}{2}i + \frac{\sqrt{3}}{2}j\right)\right)v
\]
\[
= -80\sqrt{3} \left(\frac{1}{2}i + \frac{\sqrt{3}}{2}j\right) = -40\sqrt{3}i - 120j.
\]
The magnitude of the force is
\[
|F_1| = \sqrt{(-40\sqrt{3})^2 + (-120)^2} = \sqrt{19200} \approx 138.56 \text{ pounds}.
\]
46. In this case, \(F = -125j\) and \(v\) remains the same as in Example 6.
\[
F_1 = \text{proj}_vF = (F \cdot v)v = -125 \left(\frac{\sqrt{2}}{2}\right)v = 62.5(i + j).
\]
The magnitude of the force is \(|F_1| = 62.5\sqrt{2} \approx 88.39 \text{ pounds}.

47. (a) \(v = (\cos 12^\circ)i + (\sin 12^\circ)j\)
\[
F = -2000j
\]
\[
F_1 = \text{proj}_vF = (F \cdot v)v
\]
\[
= \left((0, -2000) \cdot (\cos 12^\circ, \sin 12^\circ)\right)(\cos 12^\circ, \sin 12^\circ) = (2000 \sin 12^\circ)(\cos 12^\circ, \sin 12^\circ).
\]
Since \((\cos 12^\circ, \sin 12^\circ)\) is a unit vector, the magnitude of the force being extended is
\[
|F_1| = 2000 \sin 12^\circ \approx 415.82 \text{ pounds}.
\]
(b) We are looking for the gravitational force exerted perpendicular to the street. A unit vector perpendicular to the street is \(w = (\cos (-78^\circ), \sin (-78^\circ))\), so
\[
F_w = \text{proj}_wF = (F \cdot w)w
\]
\[
= (-2000 \sin (-78^\circ))(\cos (-78^\circ), \sin (-78^\circ)).
\]
Since \((\cos (-78^\circ), \sin (-78^\circ))\) is a unit vector, the magnitude of the force perpendicular to the street is
\(-2000 \sin (-78^\circ) \approx 1956.30 \text{ pounds}.

48. We want to determine “how much” of the 60 pound force is projected along the inclined plane.
\[
F = 60(\cos 43^\circ, \sin 43^\circ) \approx (43.88, 40.92) \text{ and } v = (\cos 18^\circ, \sin 18^\circ) \approx (0.95, 0.31)
\]
\[
\text{proj}_F = \left(\frac{(43.88, 40.92) \cdot (0.95, 0.31))}{(0.95, 0.31)}\right)(0.95, 0.31)
\]
\[
\approx 54.38 \text{ pounds. Of note, it is also possible to evaluate this problem considering the x-axis parallel to the inclined plane and the y-axis perpendicular to the plane. In this case } F = 60(\cos 25^\circ, \sin 25^\circ) \approx (54.38, 25.36).
\]
Since we only want the force in the x-direction, we immediately find our answer of 54.38 pounds.

49. Since the car weighs 2600 pounds, the force needed to lift the car is \((0, 2600)\).
\[
W = F \cdot AB = (0, 2600) \cdot (0, 5.5) = 14,300 \text{ foot-pounds}
\]
50. Since the potatoes weigh 100 pounds, the force needed to lift the potatoes is \((0, 100)\).
\[
W = F \cdot AB = (0, 100) \cdot (0, 3) = 300 \text{ foot-pounds}
\]
51. \(F = 12 \cdot \frac{(1, 2)}{|(1, 2)|} = \frac{12}{\sqrt{5}}(1, 2)
\]
\[
W = F \cdot AB = \frac{12}{\sqrt{5}}(1, 2) \cdot (4, 0) = \frac{48}{\sqrt{5}} \approx 21.47 \text{ foot-pounds}
\]
52. \(F = 24 \cdot \frac{(4, 5)}{|(4, 5)|} = \frac{24}{\sqrt{4^2 + 5^2}}(4, 5) \approx \frac{24}{\sqrt{41}}(4, 5)
\]
\[
W = F \cdot AB = \frac{24}{\sqrt{41}}(4, 5) \cdot (5, 0) = \frac{480}{\sqrt{41}} \approx 74.96 \text{ foot-pounds}
\]
53. \(F = 30 \cdot \frac{(2, 2)}{|(2, 2)|} = \frac{30}{\sqrt{2^2 + 2^2}}(2, 2) = 15\sqrt{2} (1, 1)
\]
Since we want to move 3 feet along the line \(y = \frac{1}{2}x\), we solve for \(x\) and \(y\) by using the Pythagorean theorem:
\[
x^2 + y^2 = 3^2, x^2 + \left(\frac{1}{2}x\right)^2 = 9, \frac{5}{4}x^2 = 9,
\]
\[
x = \frac{6}{\sqrt{5}}, y = \frac{3}{\sqrt{5}}
\]
\[
= -\frac{6}{\sqrt{5}}(\sqrt{5}) = -\frac{3}{5}
\]
\[
W = F \cdot AB = (15\sqrt{2}, 15\sqrt{2}) \cdot \left(\frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)
\]
\[
= 135 \cdot \frac{2}{\sqrt{5}} = 27\sqrt{50} \approx 85.38 \text{ foot-pounds}
\]
54. \(F = 50 \cdot \frac{(2, 3)}{|(2, 3)|} = \frac{50}{\sqrt{2^2 + 3^2}}(2, 3) = \frac{50}{\sqrt{13}}(2, 3)
\]
Since we want to move the object 5 feet along the line \(y = x\), we solve for \(x\) and \(y\) by using the Pythagorean theorem:
\[
x^2 + y^2 = 5^2, x^2 + x^2 = 25, 2x^2 = 25, x = \frac{25}{\sqrt{5}}, y = \frac{25}{\sqrt{5}}
\]
\[
= (2.5\sqrt{5}, 2.5\sqrt{5})
\]
\[
W = F \cdot AB = \frac{50}{\sqrt{13}}(2, 3) \cdot (2.5\sqrt{5}, 2.5\sqrt{5}) = 625 \frac{2}{\sqrt{13}} \approx 245.15 \text{ foot-pounds}
\]
55. \(W = F \cdot AB = |F| \cdot |AB| \cdot \cos \theta = 200\sqrt{13} \cdot 30^\circ
\]
\[
= 200\sqrt{13} \cdot \frac{\sqrt{3}}{2} = 100\sqrt{39} \approx 624.5 \text{ foot-pounds}
\]
56. \(AB = \langle 4, 3 \rangle - \langle -1, 1 \rangle = \langle 5, 2 \rangle
\]
\[
W = F \cdot AB = |F| \cdot |AB| \cdot \cos \theta = 75\sqrt{29} \cdot 60^\circ
\]
\[
= 75\sqrt{29} \cdot \frac{1}{2} = \frac{75 \sqrt{29}}{2} \approx 201.9 \text{ foot-pounds}
\]
57. (a) Let \( \mathbf{u} = \langle u_1, u_2 \rangle \), \( \mathbf{v} = \langle v_1, v_2 \rangle \), and \( \mathbf{w} = \langle w_1, w_2 \rangle \).
\[
\mathbf{0} \cdot \mathbf{u} = \langle 0, 0 \rangle \cdot \langle u_1, u_2 \rangle = 0 \cdot u_1 + 0 \cdot u_2 = 0
\]
(b) \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \langle u_1, u_2 \rangle \cdot (\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle) = u_1(v_1 + w_1) + u_2(v_2 + w_2) = u_1v_1 + u_2v_2 + u_1w_1 + u_2w_2 = \langle u_1, u_2 \rangle \cdot (\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}
\]

58. (a) When we evaluate the projection of \( \mathbf{u} \) onto \( \mathbf{v} \) we are actually trying to determine “how much” of \( \mathbf{u} \) is “going” in the direction of \( \mathbf{v} \). Using Figure 6.19, imagine that \( \mathbf{v} \) runs along our \( x \)-axis, with the \( y \)-axis perpendicular to it. Written in component form, \( \mathbf{u} = (|\mathbf{u}| \cos \theta, |\mathbf{u}| \sin \theta) \) and we see that the projection of \( \mathbf{u} \) onto \( \mathbf{v} \) is exactly \( |\mathbf{u}| \cos \theta \) times \( \mathbf{v} \)'s unit vector \( \frac{\mathbf{v}}{|\mathbf{v}|} \). Thus, \( \text{proj}_\mathbf{v} \mathbf{u} = |\mathbf{u}| \cos \theta \frac{\mathbf{v}}{|\mathbf{v}|} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \)

(b) Recall Figure 6.19 and let \( \mathbf{w}_1 \) be \( \mathbf{PR} = \text{proj}_\mathbf{u} \mathbf{u} \) and \( \mathbf{w}_2 = \mathbf{RQ} = \mathbf{P} - \text{proj}_\mathbf{u} \mathbf{u} \). Then, \( \mathbf{u} - \text{proj}_\mathbf{u} \mathbf{u} \cdot \left( \text{proj}_\mathbf{u} \mathbf{u} \right) = \mathbf{w}_2 \cdot \mathbf{w}_2 \). Since \( \mathbf{w}_1 \) and \( \mathbf{w}_2 \) are perpendicular, \( \mathbf{w}_1 \cdot \mathbf{w}_2 = 0 \).

59. \[
\begin{align*}
\mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2 \rangle \\
\mathbf{u} \cdot \mathbf{v} &= (u_1, u_2) \cdot (v_1, v_2) = u_1v_1 + u_2v_2
\end{align*}
\]

As the diagram indicates, the long diagonal of the parallelogram can be expressed as the vector \( \mathbf{u} + \mathbf{v} \) while the short diagonal can be expressed as the vector \( \mathbf{u} - \mathbf{v} \). The sum of the squares of the diagonals is \( |\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot -2\mathbf{v} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{u} = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{u} = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2 \), which is the sum of the squares of the sides.

60. Let \( \mathbf{u} = \langle u_1, u_2 \rangle \).
\[
\mathbf{u} \cdot \mathbf{i} + (\mathbf{u} \cdot \mathbf{j}) \mathbf{j} = (\langle u_1, u_2 \rangle \cdot \langle 1, 0 \rangle) \mathbf{i} + (\langle u_1, u_2 \rangle \cdot \langle 0, 1 \rangle) \mathbf{j} = (u_1 \mathbf{i} + u_2 \mathbf{j}) = \langle u_1, u_2 \rangle = \mathbf{u}
\]

61. False. If either \( \mathbf{u} \) or \( \mathbf{v} \) is the zero vector, then \( \mathbf{u} \cdot \mathbf{v} = 0 \) and so \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal, but they do not count as perpendicular.

62. True. \( \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = (1)^2 = 1 \).

63. \( \mathbf{u} \cdot \mathbf{v} = 0 \), so the vectors are perpendicular.

The answer is D.

64. \( \mathbf{u} \cdot \mathbf{v} = \langle 4, -5 \rangle \cdot \langle -2, -3 \rangle = 4(-2) + (-5)(-3) = -8 + 15 = 7 \)

The answer is C.

65. \( \text{proj}_\mathbf{u} \mathbf{v} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} = \left( \frac{3 + 0}{2^2} \right)(2, 0) = \left( \frac{3}{4} \right)(2, 0) = \left( \frac{3}{2}, 0 \right) \)

The answer is A.

66. The unit vector in the direction of \( \mathbf{u} \) is \( \frac{1}{\sqrt{2}} \langle -1, 1 \rangle \). The force is represented by \( 5 \) times the unit vector. The answer is B.

67. (a) \( 2 \cdot 0 + 5 \cdot 2 = 10 \) and \( 2 \cdot 5 + 5 \cdot 0 = 10 \)

(b) \( \mathbf{AP} = \langle 3 - 0, 7 - 2 \rangle = \langle 3, 5 \rangle \)
\( \mathbf{AB} = \langle 5 - 0, 0 - 2 \rangle = \langle 5, -2 \rangle \)
\( \mathbf{w}_1 = \text{proj}_{\mathbf{AB}} \mathbf{AP} = \left( \frac{3, 5 \cdot (5, -2)}{5^2 + (-2)^2} \right) \langle 5, -2 \rangle = \left( \frac{15 - 10}{29} \right) \langle 5, -2 \rangle = \left( \frac{5}{29} \right) \langle 5, -2 \rangle \)
\( \mathbf{w}_2 = \mathbf{AP} - \text{proj}_{\mathbf{AB}} \mathbf{AP} = \langle 3, 5 \rangle - \frac{5}{29} \langle 5, -2 \rangle = \langle 3 - \frac{25}{29} \cdot 5 + \frac{10}{29} \rangle = \frac{1}{29} \langle 62, 155 \rangle \)

(c) \( \mathbf{w}_2 \) is a vector from a point on \( \mathbf{AB} \) to point \( P \). Since \( \mathbf{w}_2 \) is perpendicular to \( \mathbf{AB} \), \( \mathbf{w}_2 \) is the shortest distance from \( \mathbf{AB} \) to \( P \).
\( |\mathbf{w}_2| = \sqrt{\left( \frac{62}{29} \right)^2 + \left( \frac{155}{29} \right)^2} = \sqrt{27869/29^2} = 31\sqrt{29}/29 \)

(d) Consider Figure 6.19. To find the distance from a point \( P \) to a line \( L \), we must first find \( \mathbf{u}_1 = \text{proj}_{\mathbf{AB}} \mathbf{AP} \). In this case,
\( \text{proj}_{\mathbf{AB}} \mathbf{AP} = \left( \langle x_0, y_0 - 2 \rangle \cdot \langle 5, -2 \rangle \right) \langle 5, -2 \rangle = \left( \frac{5x_0 - 2y_0 + 4}{29} \right) \langle 5, -2 \rangle = \left( \frac{25x_0 - 10y_0 + 20 - 10x_0 + 4y_0 - 8}{29} \right) \)

and
\( \mathbf{AP} - \text{proj}_{\mathbf{AB}} \mathbf{AP} = \langle x_0, y_0 - 2 \rangle = \langle 25x_0 - 10y_0 + 20 - 10x_0 + 4y_0 - 8 \rangle \)

The answer is D.
So, the distance is the magnitude of this vector.
\[
d = \frac{1}{29} \sqrt{(4x_0 + 10y_0 - 20)^2 + (10x_0 + 25y_0 - 50)^2}
\]
\[
= \frac{\sqrt{2(2x_0 + 5y_0 - 10)^2 + 5^2(2x_0 + 5y_0 - 10)^2}}{29}
\]
\[
= \frac{\sqrt{29(2x_0 + 5y_0 - 10)^2}}{29}
\]
\[
= \frac{|(2x_0 + 5y_0 - 10)|}{\sqrt{29}}
\]

(e) In the general case, \(\overrightarrow{AB} = \left( \frac{c}{a}, \frac{c}{b} \right)\) and
\[
\overrightarrow{AP} = \left( x_0, y_0 - \frac{c}{b} \right)
\]
so \(\text{proj}_{\overrightarrow{AB}} \overrightarrow{AP}\)
\[
= \left( \frac{c}{a} \right) x_0 - \left( \frac{c}{a} \right) \frac{c}{b} \]
\[
= \left( \frac{b}{a} x_0 - aby_0 + ac, \frac{a}{a^2 + b^2} \right) - \left( \frac{b}{a} x_0 - aby_0 + ac, \frac{a}{a^2 + b^2} \right)
\]
\[
= \left( \frac{a}{a^2 + b^2} - \frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2} - \frac{a}{a^2 + b^2} \right)
\]
\[
= \left( \frac{a^2 - b^2}{a^2 + b^2}, \frac{a^2 - b^2}{a^2 + b^2} \right)
\]
The magnitude of this vector, \(|\overrightarrow{AP} - \text{proj}_{\overrightarrow{AB}} \overrightarrow{AP}|\), is the distance from point \(P\) to \(L\):
\[
\frac{\left| ax_0 + by_0 - c \right|}{\sqrt{a^2 + b^2}}
\]

68. (a) Yes, if \(v = (0, 0)\) or \(t = n\pi, n = \text{any integer}\)

(b) Yes, if \(u = (0, 0)\) or \(t = \frac{n\pi}{2}, n = \text{odd integer}\)

(c) Generally, no, because \(\sin t \neq \cos t\) for most \(t\).

Exceptions, however, would occur when \(t = \frac{\pi}{4} + n\pi, n = \text{any integer}, or if } u = (0, 0) \text{ and/or } v = (0, 0)\).

69. One possible answer:
If \(au + bv = cu + dv\)
\[
au - cu + bv - dv = 0
\]
\[
(a - c)u + (b - d)v = 0
\]
Since \(u\) and \(v\) are not parallel, the only way for this equality to hold true for all vectors \(u\) and \(v\) is if \(a - c = 0\) and \(b - d = 0\), which indicates that \(a = c\) and \(b = d\).
Quick Review 6.3

1. (a) \( \overrightarrow{OA} = (-3, -2) \)
   (b) \( \overrightarrow{OB} = (4, 6) \)
   (c) \( \overrightarrow{AB} = (4 - (-3), 6 - (-2)) = (7, 8) \)

2. (a) \( \overrightarrow{OA} = (-1, 3) \)
   (b) \( \overrightarrow{OB} = (4, -3) \)
   (c) \( \overrightarrow{AB} = (4 - (-1), -3 - 3) = (5, -6) \)

3. \( m = \frac{6 - (-2)}{4 - (-3)} = \frac{8}{7} \)
   \( y + 2 = \frac{8}{7}(x + 3) \) or \( y - 6 = \frac{8}{7}(x - 4) \)

4. \( m = \frac{-3 - 3}{4 - (-1)} = \frac{-6}{5} \)
   \( y - 3 = \frac{-6}{5}(x + 1) \) or \( y + 3 = -\frac{6}{5}(x - 4) \)

5. Graph \( y = \pm \sqrt{8x} \)

6. Graph \( y = \pm \sqrt{-5x} \)

7. \( x^2 + y^2 = 4 \)
8. \( (x + 2)^2 + (y - 5)^2 = 9 \)

9. \( \frac{600 \text{ rotations}}{1 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rotation}} = 20\pi \text{ rad/sec} \)

10. \( \frac{700 \text{ rotations}}{1 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rotation}} = \frac{70}{3}\pi \text{ rad/sec} \)

Section 6.3 Exercises

1. (b) \([-5, 5]\) by \([-5, 5]\)
2. (d) \([-5, 5]\) by \([-5, 5]\)
3. (a) \([-5, 5]\) by \([-5, 5]\)
4. (c) \([-10, 10]\) by \([-12, 10]\)
5. (a) \( t \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( y )</td>
<td>(-\frac{1}{2})</td>
<td>-2</td>
<td>undefined</td>
<td>4</td>
<td>(\frac{5}{2})</td>
</tr>
</tbody>
</table>

10. \( x = 1 + y, \) so \( y = x - 1: \) line through \((0, -1)\) and \((1, 0)\)
12. \( t = y - 5 \), so \( x = 2 - 3(y - 5)\); \( y = -\frac{1}{3}x + \frac{17}{3}\), line through \( \left(0, \frac{17}{3}\right) \) and \( (17, 0)\).

13. \( t = \frac{1}{2}x + \frac{3}{2} \), so \( y = 9 - 4 \left(\frac{1}{2}x + \frac{3}{2}\right)\);
\( y = -2x + 3, 3 \leq x \leq 7\); line segment with endpoints \((3, -3)\) and \((7, -11)\).

14. \( t = y - 2 \), so \( x = 5 - 3(y - 2)\);
\( y = -\frac{1}{3}x + \frac{11}{3}, -4 \leq x \leq 8\); line segment with endpoints \((8, 1)\) and \((-4, 5)\).

15. \( x = (y - 1)^2\); parabola that opens to right with vertex at \((0, 1)\).

16. \( y = x^2 - 3\); parabola that opens upward with vertex at \((0, -3)\).

17. \( y = x^3 - 2x + 3\); cubic polynomial.

18. \( x = 2y^2 - 1\); parabola that opens to right with vertex at \((0, -1)\).

19. \( x = 4 - y^2\); parabola that opens to left with vertex at \((4, 0)\).

20. \( t = 2x\), so \( y = 16x^3 - 3\); cubic, \(-1 \leq x \leq 1\).

21. \( t = x + 3\), so
\[ y = \frac{2}{x + 3}, \text{ on domain: } [-8, -3) \cup (-3, 2]. \]

22. \( t = x - 2\), so \( y = \frac{4}{x - 2}, x \geq 4\).

23. \( x^2 + y^2 = 25\), circle of radius 5 centered at \((0, 0)\).

24. \( x^2 + y^2 = 16\), circle of radius 4 centered at \((0, 0)\).

25. \( x^2 + y^2 = 4\), three-fourths of a circle of radius 2 centered at \((0, 0)\) (not in Quadrant II).

26. \( x^2 + y^2 = 9\), semicircle of radius 3, \( y \geq 0\) only.

27. \( \overrightarrow{OA} = (-2, 5), \overrightarrow{OB} = (4, 2), \overrightarrow{OP} = (x, y)\)
\[ \overrightarrow{OP} - \overrightarrow{OA} = t(\overrightarrow{OB} - \overrightarrow{OA}) \]
\[ x + 2, y - 5 = t(6, -3) \]
\[ x + 2 = 6t \Rightarrow x = 6t - 2 \]
\[ y - 5 = 3t \Rightarrow y = -3t + 5 \]

28. \( \overrightarrow{OA} = (-3, -3), \overrightarrow{OB} = (5, 1), \overrightarrow{OP} = (x, y)\)
\[ \overrightarrow{OP} - \overrightarrow{OA} = t(\overrightarrow{OB} - \overrightarrow{OA}) \]
\[ x + 3, y + 3 = t(8, 4) \]
\[ x + 3 = 8t \Rightarrow x = 8t - 3 \]
\[ y + 3 = 4t \Rightarrow y = 4t - 3 \]

For #29–32, many answers are possible; one or two of the simplest are given.

29. Two possibilities are \( x = t + 3\),
\[ y = 4 - \frac{7}{3}t, 0 \leq t \leq 3\), or \( x = 3t + 3, y = 4 - 7t, 0 \leq t \leq 1. \)

30. Two possibilities are \( x = 5 - t, y = 2 - \frac{6}{7}t, 0 \leq t \leq 7\), or \( x = 5 - 7t, y = 2 - 6t, 0 \leq t \leq 1. \)

31. One possibility is \( x = 5 + 3 \cos t, y = 2 + 3 \sin t, 0 \leq t \leq 2\pi. \)

32. One possibility is \( x = -2 + 2 \cos t, y = -4 + 2 \sin t, 0 \leq t \leq 2\pi. \)

33. In Quadrant I, we need \( x > 0 \) and \( y > 0\), so \( 2 - |t| > 0 \) and \( t < 0.5 > 0\). Then \(-2 < t < 2\) and \( t > 0.5\), so \( 0.5 < t < 2\). This is not changed by the additional requirement that \(-3 \leq t \leq 3\).

34. In Quadrant II, we need \( x < 0 \) and \( y > 0\), so \( 2 - |t| < 0 \) and \( t < 0.5 > 0\). Then \((-2 < t < 2\) and \( t > 0.5\), so \( t > 2\). With the additional requirement that \(-3 \leq t \leq 3\), this becomes \( 2 < t \leq 3\).

35. In Quadrant III, we need \( x < 0 \) and \( y < 0\), so \( 2 - |t| < 0 \) and \( t < 0.5 < 0\). Then \((-2 < t < 2\) and \( t < 0.5\), so \( t < -2\). With the additional requirement that \(-3 \leq t \leq 3\), this becomes \(-3 \leq t < -2\).

36. In Quadrant IV, we need \( x > 0 \) and \( y < 0\), so \( 2 - |t| > 0 \) and \( t < 0.5 < 0\). Then \(-2 < t < 2\) and \( t < 0.5\), so \(-2 < t < 0.5\). This is not changed by the additional requirement that \(-3 \leq t \leq 3\).

37. (a) One good window is \([-20, 300]\) by \([-1, 10]\). If your grapher allows, use “Simultaneous” rather than “Sequential” plotting. Note that 100 yd is 300 ft. To show the whole race, use \(0 \leq t \leq 15\) (upper limit may vary), since Ben finishes in 12.916 sec. Note that it is the process of graphing (during which one observes Ben passing Jerry and crossing “the finish line” first), not the final product (which is two horizontal lines) which is needed; for that reason, no graph is shown here.

(b) After 3 seconds, Jerry is at \(20(3) = 60\) ft and Ben is at \(24(3) = 62\) ft. Ben is ahead by 2 ft.

38. (a) If your grapher allows, use “Simultaneous” rather than “Sequential” plotting. To see the whole race, use \(0 \leq t \leq 5.1\) (upper limit may vary), since the faster runner reaches the flag after 5.1 sec. Note that it is the process of graphing, not the final product (which shows a horizontal line) which is needed; for that reason, no graph is shown here.

(b) The faster runner (who is coming from the left in the simulation) arrives at \(t = 5.1\) sec. At this instant, the slower runner is 4.1 ft away from the flag; the slower runner doesn’t reach the flag until \(t = 5.5\) sec. This can be observed from the simulation, or by solving algebraically \(x_1 = 50\) and \(x_2 = 50\).

39. (a) \( y = -16t^2 + v_0t + s_0 = -16t^2 + 0t + 1000 \)
\[ = -16t^2 + 1000 \]

(b) Graph and trace: \( x = 1 \) and \( y = -16t^2 + 1000 \) with \( 0 \leq t \leq 6\), on the window \([0, 2] \) by \([0, 1200]\). Use something like 0.2 or less for Tstep. This graph will appear as a vertical line from \(1, 424\) to \(1, 1000\); it is not shown here because the simulation is accomplished by the tracing; not by the picture.

(c) When \( t = 4\), \( y = -16(4)^2 + 1000 = 744\) ft; the food containers are 744 ft above the ground after 4 sec.

40. (a) \( y = -16t^2 + v_0t + s_0 = -16t^2 + 80t + 5 \)

(b) Graph and trace: \( x = 6 \) and \( y = -16t^2 + 80t + 5 \) with \( 0 \leq t \leq 5.1\) (upper limit may vary) on \([0, 7]\) by \([0, 120]\). This graph will appear as a vertical line from about \((6, 0)\) to about \((6, 105)\). Tracing shows how the ball begins at a height of 5 ft, rises to over 100 ft, then falls back to the ground.
(e) Graph \( x = t \) and \( y = -16t^2 + 80t + 5 \) with \( 0 \leq t \leq 5.1 \) (upper limit may vary).

(d) When \( t = 4 \), \( y = -16(4)^2 + 80(4) + 5 = 69 \) ft. The ball is 69 ft above the ground after 4 sec.

(e) From the graph in (b), when \( t = 2.5 \) sec, the ball is at its maximum height of 105 ft.

41. Possible answers:
   
   (a) \( 0 < t < \frac{\pi}{2} \) (\( t \) in radians)
   
   (b) \( 0 < t < \pi \)
   
   (c) \( \frac{\pi}{2} < t < \frac{3\pi}{2} \)

42. (a) Both pairs of equations can be changed to \( x^2 + y^2 = 9 \) — a circle centered at the origin with radius 3. Also, when one chooses a point on this circle and swaps the \( x \) and \( y \) coordinates, one obtains another point on the same circle.

(b) The first begins at the right side (when \( t = 0 \)) and traces the circle counterclockwise. The second begins at the top (when \( t = 0 \)) and traces the circle clockwise.

43. (a) \( x = 400 \) when \( t \approx 2.80 \) — about 2.80 sec.

(b) When \( t \approx 2.80 \) sec, \( y \approx 7.18 \) ft.

(c) Reaching up, the outfielder's glove should be at or near the height of the ball as it approaches the wall. If hit at an angle of 20°, the ball would strike the wall about 19.74 ft up (after 2.84 sec) — the outfielder could not catch this.

44. (a) No: \( x = (120 \cos 30°)t \); this equals 350 when \( t \approx 3.37 \). At this time, the ball is at a height of \( y = -16t^2 + (120 \sin 30°)t + 4 \approx 24.59 \) ft.

(b) The ball hits the wall about 24.59 ft up when \( t \approx 3.37 \) (see part a) — not catchable.

45. (a) Yes: \( x = (5 + 120 \cos 30°)t \); this equals 350 when \( t \approx 3.21 \). At this time, the ball is at a height of \( y = -16t^2 + (120 \sin 30°)t + 4 \approx 31.59 \) ft.

(b) The ball clears the wall with about 1.59 ft to spare (when \( t \approx 3.21 \)).

46. For Linda's ball, \( x_1 = (45 \cos 44°)t \) and \( y_1 = -16t^2 + (45 \sin 44°)t + 5 \). For Chris's ball, \( x_2 = 78 - (41 \cos 39°)t \) and \( y_2 = -16t^2 + (41 \sin 39°)t + 5 \). Find (graphically) the minimum of \( d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). It occurs when \( t \approx 1.21 \) sec; the minimum distance is about 6.60 ft.

47. No: \( x = (30 \cos 70°)t \) and \( y = -16t^2 + (30 \sin 70°)t + 3 \). The dart lands when \( y = 0 \), which happens when \( t \approx 1.86 \) sec. At this point, the dart is about 19.11 ft from Tony, just over 10 in. short of the target.

48. Yes: \( x = (25 \cos 55°)t \) and \( y = -16t^2 + (25 \sin 55°)t + 4 \). The dart lands when \( y = 0 \), which happens when \( t \approx 1.45 \) sec. At this point, the dart is about 20.82 ft from Sue, inside the target.

49. The parametric equations for this motion are:

\[
x = (v + 160 \cos 20°)t
g\]

\[
y = -16t^2 + (160 \sin 20°)t + 4,
\]

where \( v \) is the velocity of the wind (in ft/sec) — it should be positive if the wind is in the direction of the hit, and negative if the wind is against the ball.

To solve this algebraically, eliminate the parameter \( t \) as follows:

\[
t = \frac{x}{v + 160 \cos 20°}
\]

So \( y = -16\left(\frac{x}{v + 160 \cos 20°}\right)^2 + 160 \sin 20°\left(\frac{x}{v + 160 \cos 20°}\right) + 4.
\]

Let \( u = \frac{400}{v + 160 \cos 20°} \), so the equation becomes:

\[-16u^2 + 54.72u - 26 = 0.
\]

Using the quadratic formula, we find that \( u = \frac{-54.72 \pm \sqrt{54.72^2 - 4(-16)(-26)}}{-32} \approx 0.57, 2.85\).

Solving 0.57 = \( \frac{400}{v + 160 \cos 20°} \) and 2.85 = \( \frac{400}{v + 160 \cos 20°} \), \( v \approx 551, v \approx -10 \). A wind speed of 551 ft/sec (375.7 mph) is unrealistic, so we eliminate that solution. So the wind will be blowing against the ball in order for the ball to hit within a few inches of the top of the wall.

To verify this graphically, graph the equation

\[
30 = -16\left(\frac{400}{v + 160 \cos 20°}\right)^2 + 160 \sin 20°\left(\frac{400}{v + 160 \cos 20°}\right) + 4,
\]

and find the zero.

50. Assuming the course is level, the ball hits the ground when \( y = -16t^2 + (180 \sin \theta) t \) equals 0, which happens when \( t = \frac{180 \sin \theta}{16} = 11.25 \sin \theta \) sec. At that time, the ball has traveled \( x = (180 \cos \theta)t \)

\[
= 2025(\cos \theta)(\sin \theta) \text{ feet.}
\]

The answers are therefore approximately:

(a) 506.25 ft.

(b) 650.82 ft.

(c) 775.62 ft.

(d) 876.85 ft.
51. \[ x = 35 \cos \left( \frac{\pi t}{6} \right) \text{ and } y = 50 + 35 \sin \left( \frac{\pi t}{6} \right) \]

52. \[ t = \frac{1}{5} x + \frac{2}{5}, \quad y = 3 + 3 \left( \frac{1}{5} x + \frac{2}{5} \right) = \frac{3}{5} x + \frac{21}{5} \]

Since \[ 3 = \frac{3}{5} (-2) + \frac{21}{5} = \frac{15}{5} = 3 \]

On the line.

53. (a) When \( t = \pi \) (or \( 3\pi \), or \( 5\pi \), etc.), \( y = 2 \). This corresponds to the highest points on the graph.

(b) The \( x \)-intercepts occur where \( y = 0 \), which happens when \( t = 0, 2\pi, 4\pi \), etc. The \( x \) coordinates at those times are (respectively) 0, 2\( \pi \), 4\( \pi \), etc., so these are \( 2\pi \) units apart.

54. (a)

(b) All 2's should be changed to 3's.

55. The particle begins at \(-10\), moves right to +2.25 (at \( t = 1.5 \)), then changes direction and returns to -4.

56. The particle begins at \(-5\), moves right to +4 (at time \( t = 2 \)), then changes direction and returns to -5.

57. The particle begins at \(-5\), moves right to about +0.07 (at time \( t \approx 0.15 \)), changes direction and moves left to about -20.81 (at time \( t \approx 4.5 \)), then changes direction and returns to -7.

58. The particle begins at \(-10\), moves right to about +0.88 (at \( t \approx 0.46 \)), changes direction and moves left to about -6.06 (at time \( t \approx 2.9 \)), then changes direction and ends at +20.

59. True. Eliminate \( t \) from the first set:

\[ t = x_1 + 1 \]
\[ y_1 = 3(x_1 + 1) + 1 \]
\[ y_1 = 3x_1 + 4 \]

Eliminate \( t \) from the second set:

\[ t = \frac{3}{2} x_2 + 2 \]
\[ y_2 = 2 \left( \frac{3}{2} x_2 + 2 \right) \]
\[ y_2 = 3x_2 + 4 \]

Both sets correspond to the rectangular equation 
\[ y = 3x + 4. \]

60. True. \( x = 0 \) and \( y = 1 \) when \( t = 1 \), and \( x = 2 \) and \( y = 5 \) when \( t = 3 \). Eliminating \( t \),

\[ t = x + 1. \]
\[ y = 2(x + 1) - 1. \]
\[ y = 2x + 1, \quad 0 \leq x \leq 2, 1 \leq y \leq 5. \]

61. \( x = (-1)^2 - 4 = -3, \quad y = -1 + \frac{1}{-1} = -2 \)

The answer is A.

62. The parametrization describes a circle of radius 2, centered at the origin and \( t \) represents the angle traveled counterclockwise from \((1, 0)\). The answer is A.

63. Set \(-16t^2 + 80t + 7 = 91\) and solve either graphically or using the quadratic formula. The answer is D.

64. The equations are both linear, so the answer is either (a), (b), or (c). Since \( t \) has a minimum value and no maximum value, the answer is C.

65. (a)

(b) \[ x^2 + y^2 = (a \cos t)^2 + (a \sin t)^2 \]
\[ = a^2 \cos^2 t + a^2 \sin^2 t \]
\[ = a^2 \]

The radius of the circles are \( a = \{1, 2, 3, 4\} \), centered at \((0, 0)\).

(c)

(d) \[ x - h = a \cos t \text{ and } y - k = a \sin t, \text{ so } \]
\[ (x - h)^2 + (y - k)^2 = (a \cos t)^2 + (a \sin t)^2 \]
\[ = a^2 \cos^2 t + a^2 \sin^2 t \]
\[ = a^2 \]

The graph is the circle of radius \( a \) centered at \((h, k)\).

(e) If \((x + 1)^2 + (y - 4)^2 = 9\), then \( a = 3, h = -1, \text{ and } k = 4. \)

As a result, \( x = 3 \cos t - 1 \) and \( y = 3 \sin t + 4. \)

66. (a)

(b)

(c) \[ t = \frac{1}{a} x - \frac{b}{a}, \quad y = c \left( \frac{1}{a} x - \frac{b}{a} \right) + d \]
\[ = \frac{c}{a} x - \frac{b - ad}{a}, a \neq 0. \]
(d) Slope: \( \frac{c}{a} \), if \( a \neq 0 \);
- y-intercept: \( \left( 0, \frac{-b + ad}{a} \right) \), if \( a \neq 0 \);
- x-intercept: \( \left( \frac{b - ad}{c}, 0 \right) \), if \( c \neq 0 \).

(e) The line will be horizontal if \( c = 0 \). The line will be vertical if \( a = 0 \).

67. (a) Jane is traveling in a circle of radius 20 feet and center \((0, 20)\), which yields \( x_1 = 20 \cos (nt) \) and \( y_1 = 20 + 20 \sin (nt) \). Since the Ferris wheel is making one revolution \((2\pi)\) every 12 seconds, \( 2\pi = 12n \), so \( n = \frac{2\pi}{12} = \frac{\pi}{6} \).

Thus,
\[
x_1 = 20 \cos \left( \frac{\pi t}{6} \right) \quad \text{and} \quad y_1 = 20 + 20 \sin \left( \frac{\pi t}{6} \right), \quad \text{in radian mode.}
\]

(b) Since the ball was released at 75 ft in the positive \(x\)-direction and gravity acts in the negative \(y\)-direction at 16 ft/s\(^2\), we have \( x_2 = at + 75 \) and \( y_2 = -16t^2 + bt \), where \( a \) is the initial speed of the ball in the \(x\)-direction and \( b \) is the initial speed of the ball in the \(y\)-direction. The initial velocity vector of the ball is \( b = 60 \) (cos 120\(^\circ\), sin 120\(^\circ\)) = \((-30, 30\sqrt{3})\), so \( a = -30 \) and \( b = 30\sqrt{3} \). As a result \( x_2 = -30t + 75 \) and \( y_2 = -16t^2 + 30\sqrt{3}t \) are the parametric equations for the ball.

(c) [Graph showing the circular path of the Ferris wheel and a line segment representing Jane and the ball's movement]

Our graph shows that Jane and the ball will be close to each other but not at the exact same point at \( t = 2.1 \) seconds.

(d) \( d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \)
\[
= \sqrt{\left( 20 \cos \left( \frac{\pi t}{6} \right) + 30t - 75 \right)^2 + \left( 20 + 20 \sin \left( \frac{\pi t}{6} \right) + 16t^2 - (30\sqrt{3})t \right)^2}
\]

(e) [Graph showing the minimum distance calculation]

The minimum distance occurs at \( t = 2.2 \), when \( d(t) = 1.64 \) feet.

68. Assuming that the bottom of the ferris wheel and the ball’s initial position are at the same height, the position of Matthew is \( x_1 = 71 \cos \left( \frac{\pi t}{10} \right) \) and \( y_1 = 71 \sin \left( \frac{\pi t}{10} \right) \). The ball’s position is \( x_2 = 90 + (88 \cos 100t) \) and \( y_2 = -16t^2 + (88 \sin 100t) \).

Find (graphically) the minimum of \( d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). It occurs when \( t \approx 2.19 \) sec; the minimum distance is about 3.47 ft.

69. Chang’s position: \( x_1 = 20 \cos \left( \frac{\pi t}{6} \right) \) and \( y_1 = 20 + 20 \sin \left( \frac{\pi t}{6} \right) \).

Kuan’s position: \( x_2 = 15 + 15 \cos \left( \frac{\pi t}{4} \right) \) and \( y_2 = 15 + 15 \sin \left( \frac{\pi t}{4} \right) \).

Find (graphically) the minimum of \( d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). It occurs when \( t \approx 21.50 \) sec; the minimum distance is about 4.11 ft.

70. Chang’s position: \( x_1 = 20 \cos \left( \frac{\pi t}{6} \right) \) and \( y_1 = 20 + 20 \sin \left( \frac{\pi t}{6} \right) \).

Kuan’s position: \( x_2 = 15 + 15 \cos \left( \frac{\pi t}{4} \right) \) and \( y_2 = 15 + 15 \sin \left( \frac{\pi t}{4} \right) \).

Find (graphically) the minimum of \( d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). It occurs when \( t \approx 12.32 \) sec; the minimum distance is about 10.48 ft.

71. (a) \( x(0) = 0c + (1 - 0)a = a \) and \( y(0) = 0d + (1 - 0)b = b \)

(b) \( x(1) = 1c + (1 - 1)a = c \) and \( y(1) = 1d + (1 - 1)b = d \)

72. \( x(0.5) = 0.5c + (1 - 0.5)a = 0.5(a + c) = (a + c)/2 \), while \( y(0.5) = (b + d)/2 \) — the correct coordinates for the midpoint.

73. Since the relationship between \( x \) and \( y \) is linear and one unit of time \( (t = 1) \) separates the two points, \( t = \frac{1}{3} \approx 0.33 \) will divide the segment into three equal pieces. Similarly, \( t = \frac{1}{4} = 0.25 \) will divide the segment into four equal pieces.

### Section 6.4 Polar Coordinates

**Exploration 1**

2. \( \left( 2, \frac{\pi}{3} \right) = (1, \sqrt{3}) \)
\( \left( -1, \frac{\pi}{2} \right) = (0, -1) \)
\( \left( 2, \pi \right) = (-2, 0) \)
\( \left( -5, \frac{3\pi}{2} \right) = (0, 5) \)
\( \left( 3, 2\pi \right) = (3, 0) \)
3. \((-1, -\sqrt{3}) = \left(-2, \frac{\pi}{3}\right)\)
\((0, 2) = \left(2, \frac{\pi}{2}\right)\)
\((3, 0) = (3, 0)\)
\((-1, 0) = (1, \pi)\)
\((0, -4) = \left(4, \frac{3\pi}{2}\right)\)

Quick Review 6.4
1. (a) Quadrant II
   (b) Quadrant III
2. (a) Quadrant I
   (b) Quadrant III
3. Possible answers: \(7\pi/4, -9\pi/4\)
4. Possible answers: \(7\pi/3, -5\pi/3\)
5. Possible answers: \(520^\circ, -200^\circ\)
6. Possible answers: \(240^\circ, -480^\circ\)
7. \((x - 3)^2 + y^2 = 4\)
8. \(x^2 + (y + 4)^2 = 9\)
9. \(a^2 = 12^2 + 10^2 - 2(12)(10)\cos 60^\circ\)
   \(a \approx 11.14\)
10. \(a^2 = 9^2 + 6^2 - 2(9)(6)\cos 40^\circ\)
    \(a \approx 5.85\)

Section 6.4 Exercises
1. \(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\)
2. \(2\sqrt{2}, 2\sqrt{2}\)
3. \((-1, -\sqrt{3})\)
4. \(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\)

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(b)

6. (a) \(\theta\)

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7. \(\frac{4\pi}{3}\)

8. \(\frac{5\pi}{6}\)

9. \(\frac{2\pi}{3}\)

10. \(\frac{17\pi}{10}\)

11. \((2, 30^\circ)\)

12. \((2, 30^\circ)\)

13. \((2, 120^\circ)\)

14. \((3, 135^\circ)\)

15. \(\frac{3}{4}, \frac{3\sqrt{3}}{4}\)

16. \(\frac{5}{4}, \frac{5\sqrt{2}}{4}\)

17. \((-2.70, 1.30)\)

18. \(1.62, 1.18)\)

19. \((2, 0)\)

20. \((0, 1)\)

21. \((0, -2)\)
Section 6.4 Polar Coordinates

22. \((-3, 0)\)
23. \(\left(2, \frac{\pi}{6} + 2n\pi\right)\) and \(\left(-2, \frac{\pi}{6} + (2n + 1)\pi\right)\),
   \(n\) an integer
24. \(\left(1, -\frac{\pi}{4} + 2n\pi\right)\) and \(\left(-1, -\frac{\pi}{4} + (2n + 1)\pi\right)\),
   \(n\) an integer
25. \((1.5, -20^\circ + 360n^\circ)\) and \((-1.5, 160^\circ + 360n^\circ)\),
   \(n\) an integer
26. \((-2.5, 50^\circ + 360n^\circ)\) and \((2.5, 230^\circ + 360n^\circ)\),
   \(n\) an integer
27. (a) \(\left(\sqrt{2}, \frac{\pi}{4}\right)\) or \(\left(-\sqrt{2}, -\frac{5\pi}{4}\right)\)
   (b) \(\left(\sqrt{2}, \frac{\pi}{4}\right)\) or \(\left(-\sqrt{2}, -\frac{3\pi}{4}\right)\)
   (c) The answers from (a), and also \(\left(\sqrt{2}, \frac{9\pi}{4}\right)\) or \(\left(-\sqrt{2}, -\frac{13\pi}{4}\right)\)
28. (a) \((\sqrt{10}, \tan^{-1} 3)\) \(\approx (\sqrt{10}, 1.25)\) or \((-\sqrt{10}, -\tan^{-1} 3 + \pi) \approx (-\sqrt{10}, 4.39)\)
   (b) \((\sqrt{10}, \tan^{-1} 3)\) \(\approx (\sqrt{10}, 1.25)\) or \((-\sqrt{10}, -\tan^{-1} 3 - \pi) \approx (-\sqrt{10}, -1.89)\)
   (c) The answers from (a), and also \((\sqrt{10}, \tan^{-1} 3 + 2\pi) \approx (\sqrt{10}, 7.53)\) or \((-\sqrt{10}, -\tan^{-1} 3 + 3\pi) \approx (-\sqrt{10}, 10.67)\)
29. (a) \((\sqrt{29}, \tan^{-1}(-2.5) + \pi) \approx (\sqrt{29}, 1.95)\) or \((-\sqrt{29}, -\tan^{-1}(-2.5) + 2\pi) \approx (-\sqrt{29}, 5.09)\)
   (b) \((-\sqrt{29}, -\tan^{-1}(-2.5)) \approx (-\sqrt{29}, -1.19)\) or \((\sqrt{29}, \tan^{-1}(-2.5) + \pi) \approx (\sqrt{29}, 1.95)\)
   (c) The answers from (a), plus \((\sqrt{29}, \tan^{-1}(-2.5) + 3\pi) \approx (\sqrt{29}, 8.23)\) or \((-\sqrt{29}, -\tan^{-1}(-2.5) + 4\pi) \approx (-\sqrt{29}, 11.38)\)
30. (a) \((-\sqrt{5}, \tan^{-1} 2) \approx (-\sqrt{5}, 1.11)\) or \((\sqrt{5}, \tan^{-1} 2 + \pi) \approx (\sqrt{5}, 4.25)\)
   (b) \((-\sqrt{5}, \tan^{-1} 2) \approx (-\sqrt{5}, 1.11)\) or \((\sqrt{5}, \tan^{-1} 2 - \pi) \approx (\sqrt{5}, -2.03)\)
   (c) The answers from (a), plus \((-\sqrt{5}, \tan^{-1} 2 + 2\pi) \approx (-\sqrt{5}, 7.39)\) or \((\sqrt{5}, \tan^{-1} 2 + 3\pi) \approx (\sqrt{5}, 10.53)\)
31. (b)
32. (d)
33. (c)
34. (a)
35. \(x = 3\) — a vertical line
36. \(y = -2\) — a horizontal line
37. \(r^2 + 3r \sin \theta = 0\), or \(x^2 + y^2 + 3y = 0\). Completing the square gives \(x^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{4}\) — a circle centered at \(\left(0, -\frac{3}{2}\right)\) with radius \(\frac{3}{2}\).
38. \(r^2 + 4r \cos \theta = 0\), or \(x^2 + y^2 + 4x = 0\). Completing the square gives \(x^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{4}\) — a circle centered at \((-2, 0)\) with radius \(2\).
39. \(r^2 - r \sin \theta = 0\), or \(x^2 + y^2 - y = 0\). Completing the square gives \(\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}\) — a circle centered at \(\left(0, \frac{1}{2}\right)\) with radius \(\frac{3}{2}\).
40. \(r^2 - 3r \cos \theta = 0\), or \(x^2 + y^2 - 3x = 0\). Completing the square gives \(\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}\) — a circle centered at \(\left(\frac{3}{2}, 0\right)\) with radius \(\frac{3}{2}\).
41. \(r^2 - 2r \sin \theta + 4r \cos \theta = 0\), or \(x^2 + y^2 - 2y + 4x = 0\). Completing the square gives \(\left(x + 2\right)^2 + (y - 1)^2 = 5\) — a circle centered at \((-2, 1)\) with radius \(\sqrt{5}\).
42. \(r^2 - 4r \cos \theta + 4r \sin \theta = 0\), or \(x^2 + y^2 - 4x + 4y = 0\). Completing the square gives \((x - 2)^2 + (y + 2)^2 = 8\) — a circle centered at \((2, -2)\) with radius \(2\sqrt{2}\).
43. \(r = 2 \cos \theta = 2 \sec \theta\) — a vertical line
44. \(r = 5 / \cos \theta = 5 \sec \theta\)
45. \(r = \frac{5}{2 \cos \theta - 3 \sin \theta}\)
46. \(r = \frac{2}{3 \cos \theta + 4 \sin \theta}\)
47. \( r^2 - 6r \cos \theta = 0 \), so \( r = 6 \cos \theta \)

48. \( r^2 - 2r \sin \theta = 0 \), so \( r = 2 \sin \theta \)

49. \( r^2 + 6r \cos \theta + 6r \sin \theta = 0 \), so \( r = -6 \cos \theta - 6 \sin \theta \)

50. \( r^2 - 2r \cos \theta + 8r \sin \theta = 0 \), so \( r = 2 \cos \theta - 8 \sin \theta \)

51. \( d = \sqrt{4^2 + 2^2 - 2 \cdot 4 \cdot 2 \cos (12^\circ - 72^\circ)} = \sqrt{20 - 16 \cos 60^\circ} = \sqrt{2} \approx 2.41 \) mi

52. \( d = \sqrt{3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cos (170^\circ - 150^\circ)} = \sqrt{34 - 30 \cos 20^\circ} \approx 2.41 \) mi

53. Using the Pythagorean theorem, the center-to-vertex distance is \( \frac{a}{\sqrt{2}} \). The four vertices are then \( \left( \frac{a}{\sqrt{2}}, \frac{\pi}{4} \right) \), \( \left( \frac{a}{\sqrt{2}}, \frac{3\pi}{4} \right) \), \( \left( \frac{a}{\sqrt{2}}, \frac{5\pi}{4} \right) \), and \( \left( \frac{a}{\sqrt{2}}, \frac{7\pi}{4} \right) \). Other polar coordinates for these points are possible, of course.

54. The vertex on the x-axis has polar coordinates \((a, 0)\). All other vertices must also be \(a\) units from the origin; their coordinates are \( \left( a, \frac{2\pi}{5} \right), \left( a, \frac{4\pi}{5} \right), \left( a, \frac{6\pi}{5} \right) \), and \( \left( a, \frac{8\pi}{5} \right) \). Other polar coordinates for these points are possible, of course.

55. False. Point \((r, \theta)\) is the same as point \((r, \theta + 2n\pi)\) for any integer \(n\). So each point has an infinite number of distinct polar coordinates.

56. True. For \((r_1, \theta)\) and \((r_2, \theta + \pi)\) to represent the same point, \((r_2, \theta + \pi)\) has to be the reflection across the origin of \((r_1, \theta + \pi)\), and this is accomplished by setting \(r_2 = -r_1\).

57. For point \((r, \theta)\), changing the sign on \(r\) and adding \(3\pi\) to \(\theta\) constitutes a twofold reflection across the origin. The answer is C.

58. The rectangular coordinates are \((-2 \cos(-\pi/3), -2 \sin(-\pi/3)) = (-1, \sqrt{3})\). The answer is C.

59. For point \((r, \theta)\), changing the sign on \(r\) and subtracting \(180^\circ\) from \(\theta\) constitutes a twofold reflection across the origin. The answer is A.

60. \((-2, 2)\) lies in Quadrant III, whereas \((-2\sqrt{2}, 135^\circ)\) lies in Quadrant IV. The answer is E.

61. (a) If \(\theta_1 - \theta_2\) is an odd integer multiple of \(\pi\), then the distance is \(|r_1 + r_2|\). If \(\theta_1 - \theta_2\) is an even integer multiple of \(\pi\), then the distance is \(|r_1 - r_2|\).

(b) Consider the triangle formed by \(O_1, P_1,\) and \(P_2\) (ensuring that the angle at the origin is less than \(180^\circ\)), then by the Law of Cosines,
\[ P_1P_2^2 = OP_1^2 + OP_2^2 - 2 \cdot OP_1 \cdot OP_2 \cos \theta, \]

where \(\theta\) is the angle between \(OP_1\) and \(OP_2\). In polar coordinates, this formula translates very nicely into
\[ d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) \text{ (or } \cos(\theta_1 - \theta_2) \text{ since } \cos(\theta_1 - \theta_2) = \cos(\theta_1 - \theta_2)), \]

so
\[ d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}. \]

(c) Yes. If \(\theta_1 - \theta_2\) is an odd integer multiple of \(\pi\), then \(\cos(\theta_1 - \theta_2) = -1 \Rightarrow d = \sqrt{r_1^2 + r_2^2 + 2r_1r_2} = |r_1 + r_2|\). If \(\theta_1 - \theta_2\) is an even integer multiple of \(\pi\), then \(\cos(\theta_1 - \theta_2) = 1 \Rightarrow d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2} = |r_1 - r_2|\).

62. (a) The right half of a circle centered at \((0, 2)\) of radius 2.

(b) Three quarters of the same circle, starting at \((0, 0)\) and moving counterclockwise.

(c) The full circle (plus another half circle found through the trace function).

(d) 4 counterclockwise rotations of the same circle.

63. \(d = \sqrt{2^2 + 2^2 - 2(2)(5) \cos 120^\circ} \approx 6.24\)

64. \(d = \sqrt{4^2 + 6^2 - 2(4)(6) \cos 45^\circ} \approx 4.25\)

65. \(d = \sqrt{(-3)^2 + (-5)^2 - 2(-3)(-5) \cos 135^\circ} \approx 7.43\)

66. \(d = \sqrt{6^2 + 8^2 - 2(6)(8) \cos 30^\circ} \approx 4.11\)

67. Since \(x = r \cos \theta\) and \(y = r \sin \theta\), the parametric equation would be \(x = f(\theta) \cos(\theta)\) and \(y = f(\theta) \sin(\theta)\).

68. \(x = 2 \cos^2 \theta\)
\(y = 2(\cos \theta) \sin \theta\)

69. \(x = 5(\cos \theta) \sin \theta\)
\(y = 5 \sin^2 \theta\)

70. \(x = 2(\cos \theta) \sec \theta = 2\)
\(y = 2(\sin \theta) \sec \theta = 2 \tan \theta\)

71. \(x = 4(\cos \theta) \csc \theta = 4 \cot \theta\)
\(y = 4(\sin \theta) \csc \theta = 4\)
Section 6.5 Graphs of Polar Equations

Exploration 1

Answers will vary.

Exploration 2

1. If \( r^2 = 4 \cos(2\theta) \), then \( r \) does not exist when \( \cos(2\theta) < 0 \). Since \( \cos(2\theta) \) is negative whenever \( \theta \) is in the interval \( \left( \frac{\pi}{4} + \pi n, \frac{3\pi}{4} + \pi n \right) \), \( n \) is any integer, the domain of \( r \) does not include these intervals.

2. \(-r\sqrt{\cos(2\theta)}\) draws the same graph, but in the opposite direction.

3. \( (r)^2 - 4 \cos(-2\theta) = r^2 - 4 \cos(2\theta) \)

4. \( (-r)^2 - 4 \cos(-2\theta) = r^2 - 4 \cos(2\theta) \)

5. \( (-r)^2 - 4 \cos(2\theta) = r^2 - 4 \cos(2\theta) \)

Quick Review 6.5

For #1–4, use your grapher’s TRACE function to solve.

1. Minimum: \(-3\) at \( x = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \); Maximum: \(3\) at \( x = \{0, \pi, 2\pi\} \)

2. Minimum: \(-1\) at \( x = \pi \); Maximum: \(5\) at \( x = \{0, 2\pi\} \)

3. Minimum: \(0\) at \( x = \{0, \pi, 2\pi\} \); Maximum: \(2\) at \( x = \{0, \pi, 2\pi\} \)

4. Minimum: \(0\) at \( x = \left\{ \frac{\pi}{2} \right\} \); Maximum: \(6\) at \( x = \frac{3\pi}{2} \)

5. (a) No (b) No (c) Yes

6. (a) No (b) Yes (c) No

7. \( \sin(\pi - \theta) = \sin \theta \)

8. \( \cos(\pi - \theta) = -\cos \theta \)

9. \( \cos(2\pi + \theta) = \cos(2\pi + 2\theta) = \cos 2\theta = \cos^2 \theta - \sin^2 \theta \)

10. \( \sin(2\pi + \theta) = \sin(2\pi + 2\theta) = \sin 2\theta = 2 \sin \theta \cos \theta \)

Section 6.5 Exercises

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(b)

\[ (0, \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\} ) \]

3. \( k = \pi \)

4. \( k = 2\pi \)

5. \( k = 2\pi \)

6. \( k = \pi \)

7. \( r_1 \) is not shown (this is a 12-petal rose). \( r_2 \) is not shown (this is a 6-petal rose). \( r_3 \) is graph (b).

8. \( 6 \cos 2\theta \sin 2\theta = 3(2 \cos u \sin u) \) where \( u = 2\theta \); this equals \( 3 \sin 2u = 3 \sin 4 \theta \). \( r = 3 \sin 4 \theta \) is the equation for the eight-petal rose shown in graph (a).
9. Graph (b) is \( r = 2 - 2 \cos \theta \): Taking \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \),
we get \( r = 2 \) and \( r = 4 \) from the first equation, and \( r = 0 \) and \( r = 2 \) from the second. No graph matches the first of these \((r, \theta)\) pairs, but (b) matches the latter (and any others one might choose).

10. Graph (c) is \( r = 2 + 3 \cos \theta \): Taking \( \theta = 0 \) and \( \theta = \pi \),
we get \( r = -1 \) from the other equation, which matches nothing. Any \((r, \theta)\) pair from the first equation matches (c), however.

11. Graph (a) is \( r = 2 - 2 \sin \theta \): where \( \theta = \pi \),
\( 2 + 2 \cos \theta = 2 \), but is clearly not on graph (a); meanwhile \( 2 - 2 \sin \frac{\pi}{2} = 0 \), and \( 0, \frac{\pi}{2} \)
(the origin) is part of graph (a).

12. Graph (d) is \( r = 2 - 1.5 \sin \theta \): where \( \theta = \pi \),
\( 2 + 1.5 \cos \theta = 2 \), but is clearly not on graph (d); meanwhile \( 2 - 1.5 \sin \frac{\pi}{2} = 0.5 \), and
\( 0.5, \frac{\pi}{2} \) is part of graph (d).

13. Symmetric about the y-axis: replacing \((r, \theta)\) with \((r, \pi - \theta)\) gives the same equation, since \( \sin(\pi - \theta) = \sin \theta \).

14. Symmetric about the x-axis: replacing \((r, \theta)\) with \((r, -\theta)\) gives the same equation, since \( \cos(-\theta) = \cos \theta \).

15. Symmetric about the y-axis: replacing \((r, \theta)\) with \((r, -\theta)\) gives the same equation, since \( \sin(-\theta) = \sin \theta \).

16. Symmetric about the x-axis: replacing \((r, \theta)\) with \((r, \pi + \theta)\) gives the same equation, since \( \cos(2\theta + \pi) = \cos(2\theta + 2\pi) = \cos 2\theta \).

17. All three symmetries. Polar axis: replacing \((r, \theta)\) with \((r, -\theta)\) gives the same equation, since \( \cos(-2\theta) = \cos 2\theta \).

18. Symmetric about the y-axis: replacing \((r, \theta)\) with \((-r, -\theta)\) gives the same equation, since \( \sin(-3\theta) = -\sin 3\theta \).

19. Symmetric about the y-axis: replacing \((r, \theta)\) with \((r, \pi - \theta)\) gives the same equation, since \( \sin(\pi - \theta) = \sin \theta \).

20. Symmetric about the x-axis: replacing \((r, \theta)\) with \((r, -\theta)\) gives the same equation, since \( \cos(-\theta) = \cos \theta \).

21. Maximum \( r \) is 5 — when \( \theta = 2n\pi \) for any integer \( n \).

22. Maximum \( r \) is 5 (actually, -5) — when \( \theta = \frac{3\pi}{2} + 2n\pi \) for any integer \( n \).

23. Maximum \( r \) is 3 (along with -3) — when \( \theta = 2n\pi/3 \) for any integer \( n \).

24. Maximum \( r \) is 4 (along with -4) — when \( \theta = n\pi/3 \) for any odd integer \( n \).
29. Domain: All reals
   Range: \([-2, 2]\]
   Symmetric about the y-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 2
   No asymptotes

30. Domain: All reals
    Range: \([-3, 3]\]
    Symmetric about the \(x\)-axis, \(y\)-axis, and origin
    Continuous
    Bounded
    Maximum \(r\)-value: 3
    No asymptotes

31. Domain: All reals
    Range: \([1, 9]\]
    Symmetric about the y-axis
    Continuous
    Bounded
    Maximum \(r\)-value: 9
    No asymptotes

32. Domain: All reals
    Range: \([2, 11]\]
    Symmetric about the \(y\)-axis
    Continuous
    Bounded
    Maximum \(r\)-value: 11
    No asymptotes

33. Domain: All reals
    Range: \([0, 8]\]
    Symmetric about the \(x\)-axis
    Continuous
    Bounded
    Maximum \(r\)-value: 8
    No asymptotes

34. Domain: All reals
    Range: \([0, 10]\]
    Symmetric about the \(y\)-axis
    Continuous
    Bounded
    Maximum \(r\)-value: 10
    No asymptotes

35. Domain: All reals
    Range: \([3, 7]\]
    Symmetric about the \(x\)-axis
    Continuous
    Bounded
    Maximum \(r\)-value: 7
    No asymptotes

36. Domain: All reals
    Range: \([2, 4]\]
    Symmetric about the \(y\)-axis
    Continuous
    Bounded
    Maximum \(r\)-value: 4
    No asymptotes
37. Domain: All reals
   Range: \([-3, 7]\)
   Symmetric about the \(x\)-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 7
   No asymptotes

38. Domain: All reals
   Range: \([-1, 7]\)
   Symmetric about the \(y\)-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 7
   No asymptotes

39. Domain: All reals
   Range: \([0, 2]\)
   Symmetric about the \(x\)-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 2
   No asymptotes

40. Domain: All reals
   Range: \([1, 3]\)
   Symmetric about the \(y\)-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 3
   No asymptotes

41. Domain: All reals
   Range: \([0, \infty)\)
   Continuous
   No symmetry
   Unbounded
   Maximum \(r\)-value: none
   No asymptotes
   Graph for \(\theta \geq 0\):

42. Domain: All reals
   Range: \([0, \infty)\)
   Continuous
   No symmetry
   Unbounded
   Maximum \(r\)-value: none
   No asymptotes
   Graph for \(\theta \geq 0\):

43. Domain: \([0, \frac{\pi}{2}] \cup \left[\frac{3\pi}{2}, 2\pi\right]\)
   Range: \([0, 1]\)
   Symmetric about the origin
   Continuous on each interval in domain
   Bounded
   Maximum \(r\)-value: 1
   No asymptotes

44. Domain: \([0, \frac{\pi}{4}] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right]\)
   Range: \([0, 3]\)
   Symmetric about the \(x\)-axis, \(y\)-axis, and origin
   Continuous on each interval in domain
   Bounded
   Maximum \(r\)-value: 3
   No asymptotes
For #45–48, recall that the petal length is the maximum \( r \)-value over the interval that creates the petal.

45. \( r = -2 \) when \( \theta = \left\{ \frac{3\pi}{4}, \frac{7\pi}{4} \right\} \) and \( r = 6 \) when
\[
\theta = \left\{ \frac{\pi}{4}, \frac{5\pi}{4} \right\}.
\]
There are four petals with lengths \([6, 2, 6, 2]\).

46. \( r = -2 \) when \( \theta = \{0, \pi\} \) and \( r = 8 \) when
\[
\theta = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}.
\]
There are four petals with lengths \([2, 8, 2, 8]\).

47. \( r = -3 \) when \( \theta = \left\{ 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5} \right\} \) and
\[
r = 5 \text{ when } \theta = \left\{ \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5} \right\}.
\]
There are ten petals with lengths \([3, 5, 3, 5, 3, 5, 3, 5]\).

48. \( r = 7 \) when \( \theta = \left\{ \frac{\pi}{10}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10} \right\} \) and
\[
r = -1 \text{ when } \theta = \left\{ \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{11\pi}{10}, \frac{13\pi}{10} \right\}.
\]
There are ten petals with lengths \([7, 1, 7, 1, 7, 1, 7, 1]\).

49. \( r_1 \) and \( r_2 \) produce identical graphs — \( r_1 \) begins at \((1, 0)\) and \( r_2 \) begins at \((-1, 0)\).

50. \( r_1 \) and \( r_2 \) produce identical graphs — \( r_1 \) begins at \((3, 0)\) and \( r_2 \) begins at \((1, 0)\).

51. \( r_2 \) and \( r_3 \) produce identical graphs — \( r_1 \) begins at \((3, 0)\) and \( r_3 \) begins at \((-3, 0)\).

52. \( r_1 \) and \( r_2 \) produce identical graphs — \( r_1 \) begins at \((2, 0)\) and \( r_2 \) begins at \((-2, 0)\).

53. (a) A 4-petal rose curve with 2 short petals of length 1 and 2 long petals of length 3.
   (b) Symmetric about the origin.
   (c) Maximum \( r \)-value: 3.

54. (a) A 4-petal rose curve with petals of about length 1, 3, 3, and 4 units.
   (b) Symmetric about the \( y \)-axis.
   (c) Maximum \( r \)-value: 4.

55. (a) A 6-petal rose curve with 3 short petals of length 2 and 3 long petals of length 4.
   (b) Symmetric about the \( x \)-axis.
   (c) Maximum \( r \)-value: 4.

56. (a) A 6-petal rose curve with 3 short petals of length 2 and 3 long petals of length 4.
   (b) Symmetric about the \( y \)-axis.
   (c) Maximum \( r \)-value: 4.

57. Answers will vary but generally students should find that \( a \) controls the length of the rose petals and \( n \) controls both the number of rose petals and symmetry. If \( n \) is odd, \( n \) rose petals are formed, with the cosine curve symmetric about the polar \( x \)-axis and sine curve symmetric about the \( y \)-axis. If \( n \) is even, \( 2n \) rose petals are formed, with both the cosine and sine functions having symmetry about the polar \( x \)-axis, \( y \)-axis, and origin.

58. Symmetry about \( y \)-axis: \( r - 3 \sin(4\theta) = 0 \Rightarrow -r - 3 \sin(-4\theta) = r - 3 \sin(4\theta) \) (since \( \sin(\theta) \) is odd, i.e., \( \sin(-\theta) = -\sin(\theta) \)) \( r - 3 \sin(4\theta) = 0 \). Symmetry about the origin: \( r - 3 \sin(4\theta) = 0 \Rightarrow -r - 3 \sin(4\theta + 4\pi) = r - 3 \sin(4\theta) = 0 \).

59. \( y = 3 - 3 \sin x \) has minimum and maximum values of 0 and 6 on \([0, 2\pi]\). So the range of the polar function \( r = 3 - 3 \sin \theta \) is also \([0, 6]\).

60. \( y = 2 + 3 \cos x \) has minimum and maximum values of \(-1\) and 5 on \([0, 2\pi]\). So the range of the polar function \( r = 2 + 3 \cos \theta \) is also \([-1, 5]\).

In general, this works because any polar graph can also be plotted using rectangular coordinates. Here, we have \( y \) representing \( r \) and \( x \) representing \( \theta \) on a rectangular coordinate graph. Since \( y \) is exactly equal to \( r \), the range of \( y \) and range of \( r \) will be exactly the same.

61. False. The spiral \( r = \theta \) is unbounded, since a point on the curve can be found at any arbitrarily large distance from the origin by setting \( \theta \) numerically equal to that distance.

62. True. If point \((r, \theta)\) satisfies the equation \( r = 2 + \cos \theta \), then point \((r, \theta)\) does also, since \( 2 + \cos(-\theta) = 2 + \cos \theta = r \).

63. With \( r = a \cos n\theta \), if \( n \) is even there are \( 2n \) petals. The answer is \( D \).

64. The four petals lie along the \( x \)- and \( y \)-axis, because \( \cos 2\theta \) takes on its extreme values at multiples of \( \pi/2 \). The answer is \( D \).

65. When \( \cos \theta = -1, r = 5 \). The answer is \( B \).

66. With \( r = a \sin n\theta \), if \( n \) is odd there are \( n \) petals. The answer is \( B \).

67. (a) Symmetry about the polar \( x \)-axis: \( r - a \cos (n\theta) = 0 \Rightarrow -r - a \cos (-n\theta) = r - a \cos (n\theta) \) (since \( \cos(\theta) \) is even, i.e., \( \cos(-\theta) = \cos(\theta) \) for all \( \theta \)) = 0.
   (b) No symmetry about \( y \)-axis: \( r - a \cos (n\theta) = 0 \Rightarrow -r - a \cos (-n\theta) = -r - a \cos (n\theta) \) (since \( \cos(\theta) \) is even) \( \neq r - a \cos (n\theta) \) unless \( r = 0 \). As a result, the equation is not symmetric about the \( y \)-axis.
   (c) No symmetry about origin: \( r - a \cos (n\theta) = 0 \Rightarrow -r - a \cos (n\theta) \neq r - a \cos (n\theta) \) unless \( r = 0 \). As a result, the equation is not symmetric about the origin.
   (d) Since \(|\cos(n\theta)| \leq 1\) for all \( \theta \), the maximum \( r \)-value is \(|a|\).
(e) Domain: All reals
Range: $[-|a|, |a|]
Symmetric about the $x$-axis
Continuous
Bounded
Maximum $r$-value: 2
No asymptotes

68. (a) Symmetry about the $y$-axis: $r - a \sin (n\theta) = 0 \Rightarrow -r - a \sin (-n\theta) = r + a \sin (n\theta)$ (since $\sin (\theta)$ is odd, $\sin (-\theta) = -\sin (\theta)$) $= -1(r - a \sin (n\theta))$

(b) Not symmetric about polar $x$-axis: $r - a \sin (n\theta) = 0 \Rightarrow r - a \sin (-n\theta) = r + a \sin (n\theta)$. The two functions are equal only when $-\sin (n\theta) = \sin (n\theta) = 0$, or $\theta = \{0, \pi\}$, so $r - a \sin (n\theta)$ is not symmetric about the polar $x$-axis.

(c) Not symmetric about origin: $r - a \sin (n\theta) \Rightarrow -r - a \sin (n\theta) = -(r + a \sin (n\theta))$. The two functions are equal only when $r = 0$, so $r - a \sin (n\theta)$ is not symmetric about the origin.

(d) Since $|\sin (n\theta)| \leq 1$ for all $\theta$, the maximum $r$-value is $|a|$.

(e) Domain: All reals
Range: $[-|a|, |a|]
Symmetric about $y$-axis
Continuous
Bounded
Maximum $r$-value: $|a|
No asymptotes

69. (a) For $r_1$: $0 \leq \theta \leq 4\pi$ (or any interval that is $4\pi$ units long). For $r_2$: same answer.
(b) $r_1$: 10 (overlapping) petals. $r_2$: 14 (overlapping) petals.

70. Starting with the graph of $r_1$, if we rotate clockwise (centered at the origin) by $\pi/4$ radians ($45^\circ$), we get the graph of $r_2$; rotating $r_1$ counterclockwise by $\pi/3$ radians ($60^\circ$) gives the graph of $r_3$.

71. Starting with the graph of $r_1$, if we rotate counterclockwise (centered at the origin) by $\pi/4$ radians ($45^\circ$), we get the graph of $r_2$; rotating $r_1$ counterclockwise by $\pi/3$ radians ($60^\circ$) gives the graph of $r_3$. 

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(a) $[-3, 3] \times [-3, 3]$
(b) $[-3, 3] \times [-3, 3]$
(c) $[-3, 3] \times [-3, 3]$

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(a) $[-5, 5] \times [-5, 5]$
(b) $[-5, 5] \times [-5, 5]$
(c) $[-5, 5] \times [-5, 5]$
Section 6.6 DeMoivre’s Theorem and \( n \)th Roots

Quick Review 6.6

1. Using the quadratic equation to find the roots of \( x^2 + 13 = 4x \), we have \( x^2 - 4x + 13 = 0 \) with \( a = 1 \), \( b = -4 \), and \( c = 13 \).

\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}
\]

\[
x = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}
\]

\[
x = \frac{4 + 6i}{2} \quad \text{and} \quad x = \frac{4 - 6i}{2}
\]

The roots are 2 + 3i and 2 - 3i.

2. Using the quadratic equation to find the roots of \( 5(x^2 + 1) = 6x \), we have \( 5x^2 - 6x + 5 = 0 \) with \( a = 5 \), \( b = -6 \), and \( c = 5 \).

\[
x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(5)}}{2(5)}
\]

\[
x = \frac{6 \pm \sqrt{36 - 100}}{10} = \frac{6 \pm \sqrt{-64}}{10} = \frac{6 \pm 8i}{10}
\]

\[
x = \frac{6 + 8i}{10} = \frac{6}{10} + \frac{8i}{10} = 0.6 + 0.8i \quad \text{and} \quad x = \frac{6 - 8i}{10} = \frac{6}{10} + \frac{-8i}{10} = 0.6 - 0.8i
\]

The roots are 0.6 + 0.8i and 0.6 - 0.8i.

3. \((1 + i)^3 = (1 + i) \cdot [(1 + i)^2] = (1 + i) \cdot (2i)^2 = -4(1 + i) = -4 - 4i\)

4. \((1 - i)^4 = [(1 - i)^2]^2 = (-2i)^2 = -4 = -4 + 0i\)

For #5–8, use the given information to find a point \( P \) on the terminal side of the angle, which in turn determines the quadrant of the terminal side.

5. \(P(-\sqrt{3}, 1)\), in quadrant II: \( \theta = \frac{5\pi}{6} \)

6. \(P(1, -1)\), in quadrant IV: \( \theta = \frac{7\pi}{4} \)

7. \(P(-1, -\sqrt{3})\), in quadrant III: \( \theta = \frac{4\pi}{3} \)

8. \(P(-1, 1)\), in quadrant IV: \( \theta = \frac{5\pi}{4} \)

9. \(x^3 = 1\) when \( x = 1 \)

10. \(x^4 = 1\) when \( x = \pm 1 \)
3. \( r = |3i| = 3 \); \( \cos \theta = 0 \) and \( \sin \theta = 1 \), so \( \theta = \frac{\pi}{2} \\
3i = 3 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \)

4. \( r = |-2i| = 2 \); \( \cos \theta = 0 \) and \( \sin \theta = -1 \), so \( \theta = \frac{3\pi}{2} \\
-2i = 2 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \)

5. \( r = |2 + 2i| = 2\sqrt{2} \); \( \cos \theta = \frac{\sqrt{2}}{2} \) and \( \sin \theta = \frac{\sqrt{2}}{2} \), 
so \( \theta = \frac{\pi}{4} \); \( 2 + 2i = 2\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \)

6. \( r = |\sqrt{3} + i| = 2 \); \( \cos \theta = \frac{\sqrt{3}}{2} \) and \( \sin \theta = \frac{1}{2} \), 
so \( \theta = \frac{\pi}{6} \); \( \sqrt{3} + i = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \)

7. \( r = |-2 + 2\sqrt{3}| = 4 \); \( \cos \theta = -\frac{1}{2} \) and \( \sin \theta = \frac{\sqrt{3}}{2} \), 
so \( \theta = \frac{2\pi}{3} \); \(-2 + 2i\sqrt{3} = 4 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \)

8. \( r = |3 - 3i| = 3\sqrt{2} \); \( \cos \theta = -\frac{\sqrt{2}}{2} \) and \( \sin \theta = \frac{\sqrt{2}}{2} \), 
so \( \theta = \frac{7\pi}{4} \); \( 3 - 3i = 3\sqrt{2} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) \)

9. \( r = |\sqrt{3} + 2i| = \sqrt{13} \); \( \cos \theta = \frac{3}{\sqrt{13}} \) and 
\( \sin \theta = \frac{2}{\sqrt{13}} \), so \( \theta \approx 0.588 \): 
\( 3 + 2i \approx \sqrt{13} \left( \cos 0.588 + i \sin 0.588 \right) \)

10. \( r = |4 - 7i| = \sqrt{65} \); \( \cos \theta = \frac{4}{\sqrt{65}} \) and 
\( \sin \theta = -\frac{7}{\sqrt{65}} \), so \( \theta \approx 5.232 \): 
\( 4 - 7i \approx \sqrt{65} \left( \cos 5.23 + i \sin 5.23 \right) \)

11. \( r = 3; 30^\circ = \frac{\pi}{6} \); 
\( 3 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \)

12. \( r = 3; 225^\circ = \frac{5\pi}{4} \); 
\( 4 \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) \)

13. \( \frac{\sqrt{3}}{2} - \frac{1}{2}i = \frac{3}{2}\sqrt{3} - \frac{3}{2}i \)

14. \( -\frac{\sqrt{3}}{2} - \frac{1}{2}i = -4\sqrt{3} - 4i \)

15. \( \frac{1}{2} - \frac{\sqrt{3}}{2}i = 5 \frac{5}{2} - \frac{5}{2}\sqrt{3}i \)

16. \( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = \frac{5}{2}\sqrt{2} + \frac{5}{2}\sqrt{2}i \)

17. \( \sqrt{2} \left( \frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i \)

18. \( \approx 2.56 + 0.68i \)

19. \((2 \cdot 7)[\cos(25^\circ + 130^\circ) + i \sin(25^\circ + 130^\circ)] = 14 \left( \cos 155^\circ + i \sin 155^\circ \right) \)

20. \((\sqrt{2} \cdot 0.5)[\cos(188^\circ - 19^\circ) + i \sin(118^\circ - 19^\circ)] = \frac{\sqrt{2}}{2} \left( \cos 99^\circ + i \sin 99^\circ \right) \)

21. \((5 \cdot 3) \left[ \cos \left( \frac{\pi}{4} + \frac{5\pi}{3} \right) + i \sin \left( \frac{\pi}{4} + \frac{5\pi}{3} \right) \right] = 15 \left( \cos \frac{23\pi}{12} + i \sin \frac{23\pi}{12} \right) \)

22. \((\sqrt{3} \cdot \frac{1}{3}) \left[ \cos \left( \frac{3\pi}{4} + \frac{\pi}{6} \right) + i \sin \left( \frac{3\pi}{4} + \frac{\pi}{6} \right) \right] = \frac{\sqrt{3}}{3} \left( \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right) \)

23. \(\frac{2}{3} \left[ \cos(30^\circ - 60^\circ) + i \sin(30^\circ - 60^\circ) \right] = \frac{2}{3} \left( \cos(-30^\circ) + i \sin(-30^\circ) \right) = \frac{2}{3} \left( \cos 30^\circ - i \sin 30^\circ \right) \)

24. \(\frac{5}{2} \left[ \cos(220^\circ - 115^\circ) + i \sin(220^\circ - 115^\circ) \right] = \frac{5}{2} \left( \cos 105^\circ + i \sin 105^\circ \right) \)

25. \(\frac{6}{3} \left[ \cos(5\pi - 2\pi) + i \sin(5\pi - 2\pi) \right] = 2 \left( \cos 3\pi + i \sin 3\pi \right) = 2 \left( \cos \pi + i \sin \pi \right) \)

26. \(1 \left[ \cos \left( \frac{\pi}{2} - \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{2} - \frac{\pi}{4} \right) \right] = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \)

27. \(a) 3 - 2i \approx \sqrt{13} \left[ \cos(5.695) + i \sin(5.695) \right] \)

and \(1 + i = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \), so 
\(\sqrt{13} \left[ \cos(5.695) + i \sin(5.695) \right] \cdot \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 5 + i \)

\(\frac{\sqrt{2} \left[ \cos(5.695) + i \sin(5.695) \right]}{\sqrt{2} \left[ \cos(\pi/4) + i \sin(\pi/4) \right]} \approx \sqrt{65} \left[ \cos \left( \frac{5.695}{\pi - \frac{4}{\pi}} \right) + i \sin \left( \frac{5.695}{\pi - \frac{4}{\pi}} \right) \right] \)

\(= \frac{1}{2} - \frac{5}{2}i \)

(b) \((3 - 2i)(1 + i) = 3 + 3i - 2i - 2i^2 = 5 + i \)

\(\frac{3 - 2i}{1 + i} = \frac{3 - 2i}{1 + i} \left( \frac{1 - i}{1 - i} \right) = \frac{1 - 5i}{2} = \frac{1}{2} - \frac{5}{2}i \)
28. (a) \(1 - i = \sqrt{2} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right)\)
and \(\sqrt{3} + i = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)\), so
\[\sqrt{2} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) \cdot 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = 2 \sqrt{2} \left( \cos \frac{23\pi}{12} + \sin \frac{23\pi}{12} \right) \approx 2.73 - 0.73i\]
\[\sqrt{2} \approx 1.41 \left( \cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12} \right) \approx 0.18 - 0.68i\]
(b) \((1 - i)(\sqrt{3} + i) = \sqrt{3} + i - \sqrt{3}i - i^2\)
\[1 + \sqrt{3} = \frac{1 - i}{\sqrt{3} + i} = \frac{1 - i}{\sqrt{3} - i} = \frac{(1 - i)(\sqrt{3} - i)}{4} \approx 0.18 - 0.68i\]

29. (a) \(3 + i \approx \sqrt{10} \left[ \cos(0.321) + i \sin(0.321) \right]\)
and \(5 - 3i \approx \sqrt{34} \left[ \cos(-0.540) + i \sin(-0.540) \right]\), so
\[\sqrt{10} \left[ \cos(0.321) + i \sin(0.321) \right] \cdot \sqrt{34} \left[ \cos(-0.540) + i \sin(-0.540) \right] = 2 \sqrt{85} \left[ \cos(-0.219) + i \sin(-0.219) \right] = 18 - 4i\]
\[\sqrt{10} \left[ \cos(0.321) + i \sin(0.321) \right] \approx 0.35 + 0.41i\]
(b) \((3 + i)(5 - 3i) = 15 - 9i + 5i - 3i^2 = 18 - 4i\)
\[
\begin{align*}
3 + i & 
\frac{3 + i}{5 - 3i} = \frac{5 - 3i}{5 - 3i} = \frac{(3 + i)(5 - 3i)}{34} \\
1/17 & 
\approx 0.35 + 0.41i
\end{align*}
\]

30. (a) \(2 - 3i \approx \sqrt{13} \left[ \cos(-0.982) + i \sin(-0.982) \right]\)
and \(1 - \sqrt{3}i = 2 \left[ \cos \left( -\frac{\pi}{3} \right) + i \sin \left( -\frac{\pi}{3} \right) \right]\), so
\[
\begin{align*}
\sqrt{13} \left[ \cos(-0.982) + i \sin(-0.982) \right] & 
\left[ \cos \left( -\frac{\pi}{3} \right) + i \sin \left( -\frac{\pi}{3} \right) \right] \\
& 
2 \left[ \cos \left( -\frac{\pi}{3} \right) \right] + i \sin \left( -\frac{\pi}{3} \right) \\
& 
-3.20 - 6.46i \\
& 
\approx 0.18 + 0.12i
\end{align*}
\]
(b) \((2 - 3i)(1 - \sqrt{3}i) = 2 - 2\sqrt{3}i - 3i + 3\sqrt{3}i^2\)
\[= (2 - 3\sqrt{3}i + 2\sqrt{3}i + 3i) \approx -3.196 - 6.46i\]
\[1 - \sqrt{3}i = \frac{1 - \sqrt{3}i}{1 - \sqrt{3}i + 1 + \sqrt{3}i} = \frac{(2 - 3i)(1 + \sqrt{3}i)}{4} \approx 1.80 + 0.12i\]

31. \((\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})^3 = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\)
\[= -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\]
32. \(3 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)^5 = 243 \left( \cos \frac{15\pi}{2} + i \sin \frac{15\pi}{2} \right) = 243i\]
33. \(2 \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)^3 = 8 \left( \cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4} \right) = 4 \sqrt{2} + 4 \sqrt{2}i\]
34. \(6 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)^4 = 1296 \left( \cos \frac{20\pi}{6} + i \sin \frac{20\pi}{6} \right) = -648 - 648\sqrt{3}\]
35. \((1 + i)^3 = \left[ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^3 = (\sqrt{2})^3 \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = 4 \sqrt{2} \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = -8 - 4i\]
36. \((3 + 4i)^{20} = 5^{20} \left[ \cos \left( \frac{4\pi}{3} \right) + i \sin \left( \frac{4\pi}{3} \right) \right] = 5^{20} \left[ \cos \left( \frac{20\pi}{3} \right) + i \sin \left( \frac{20\pi}{3} \right) \right] = 5^{20} \left[ \cos \left( 5.979 \right) + i \sin \left( 5.979 \right) \right] \approx 5^{20} \left( 0.95 - 0.30i \right)\]
37. \((1 - \sqrt{3}i)^3 = 2 \left[ \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right]^3 = 8(\cos 5\pi + i \sin 5\pi) = 8(\cos 5\pi + i \sin 5\pi) = -8\]
38. \(\left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right)^3 = \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^3 = \cos \pi + i \sin \pi = -1\]

For #39-44, the cube roots of \(r(\cos \theta + i \sin \theta)\) are
\[= \sqrt[3]{\cos \theta + 2k\pi + i \sin \theta + 2k\pi}, \quad k = 0, 1, 2\]
39. \[\sqrt[3]{ \cos \left( \frac{2k\pi + \pi}{3} + \frac{i \sin 2k\pi + \pi}{3} \right) } = \sqrt[3]{ \cos \left( \frac{2\pi(k+1)}{3} + i \sin \frac{2\pi(k+1)}{3} \right) }, \quad k = 0, 1, 2\]
40. \[\sqrt[3]{ \cos \left( \frac{2k\pi + \pi}{3} + i \sin \frac{2k\pi + \pi}{3} \right) } = \sqrt[3]{ \cos \left( \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) }, \quad k = 0, 1, 2\]
\[ \sqrt{2} \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right), \]
\[ \sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = -1 + i \sqrt{2}. \]
\[ \sqrt{2} \left( \cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right). \]

41. \[ \sqrt[3]{\cos \frac{2k\pi + 4\pi/3}{3} + i \sin \frac{2k\pi + 4\pi/3}{3}} = \sqrt[3]{\cos \frac{2\pi(3k + 2)}{9} + i \sin \frac{2\pi(3k + 2)}{9}}, \]
\[ k = 0, 1, 2: \]
\[ \sqrt[3]{\cos \frac{4\pi}{9} + i \sin \frac{4\pi}{9}}, \sqrt[3]{\cos \frac{10\pi}{9} + i \sin \frac{10\pi}{9}}, \sqrt[3]{\cos \frac{16\pi}{9} + i \sin \frac{16\pi}{9}}. \]

42. \[ \sqrt[3]{\cos \frac{2k\pi + 11\pi/6}{3} + i \sin \frac{2k\pi + 11\pi/6}{3}} = \sqrt[3]{\cos \frac{\pi(12k + 11)}{18} + i \sin \frac{\pi(12k + 11)}{18}}, \]
\[ k = 0, 1, 2: \]
\[ \sqrt[3]{\cos \frac{11\pi}{18} + i \sin \frac{11\pi}{18}}, \sqrt[3]{\cos \frac{23\pi}{18} + i \sin \frac{23\pi}{18}}, \sqrt[3]{\cos \frac{35\pi}{18} + i \sin \frac{35\pi}{18}}. \]

43. \[ \sqrt[5]{\cos \frac{2k\pi + 5.355}{3} + i \sin \frac{2k\pi + 5.355}{3}} = \sqrt[5]{\cos \frac{\pi(8k + 1)}{12} + i \sin \frac{\pi(8k + 1)}{12}}, \]
\[ k = 0, 1, 2, \]
\[ \approx \sqrt[5]{\cos (1.79 + i \sin 1.79)}, \approx \sqrt[5]{\cos (3.88 + i \sin 3.88)}, \approx \sqrt[5]{\cos (5.97 + i \sin 5.97)}. \]

44. \[ -2 + 2i = 2 \sqrt[3]{\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}}. \]
Note that \[ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 1 + i \sqrt{2}. \]

45. \[ \cos \frac{2k\pi + \pi}{5} + i \sin \frac{2k\pi + \pi}{5} = \cos \frac{\pi(2k + 1)}{5} + i \sin \frac{\pi(2k + 1)}{5}, \]
\[ k = 0, 1, 2, 3, 4. \]

46. \[ \sqrt[3]{2} \left( \cos \frac{2k\pi + \pi/2}{5} + i \sin \frac{2k\pi + \pi/2}{5} \right) = 2 \left( \cos \frac{\pi(4k + 1)}{10} + i \sin \frac{\pi(4k + 1)}{10} \right), \]
\[ k = 0, 1, 2, 3, 4: \]
\[ 2 \left( \cos \frac{9\pi}{10} + i \sin \frac{9\pi}{10} \right), 2 \left( \cos \frac{13\pi}{10} + i \sin \frac{13\pi}{10} \right). \]

47. \[ \sqrt[3]{2} \left( \cos \frac{2k\pi + \pi/6}{5} + i \sin \frac{2k\pi + \pi/6}{5} \right) = 2 \left( \cos \frac{\pi(12k + 1)}{15} + i \sin \frac{\pi(12k + 1)}{15} \right), \]
\[ k = 0, 1, 2, 3, 4: \]
\[ \sqrt[3]{2} \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right), \sqrt[3]{2} \left( \cos \frac{3\pi}{2} + i \sin \frac{5\pi}{3} \right), \sqrt[3]{2} \left( \cos \frac{9\pi}{10} + i \sin \frac{9\pi}{10} \right), \sqrt[3]{2} \left( \cos \frac{13\pi}{10} + i \sin \frac{13\pi}{10} \right). \]

48. \[ \sqrt[3]{2} \left( \cos \frac{2k\pi + \pi/4}{5} + i \sin \frac{2k\pi + \pi/4}{5} \right) = 2 \left( \cos \frac{\pi(8k + 1)}{10} + i \sin \frac{\pi(8k + 1)}{10} \right), \]
\[ k = 0, 1, 2, 3, 4: \]
\[ \sqrt[3]{2} \left( \cos \frac{\pi}{20} + i \sin \frac{\pi}{20} \right), \sqrt[3]{2} \left( \cos \frac{9\pi}{20} + i \sin \frac{9\pi}{20} \right), \sqrt[3]{2} \left( \cos \frac{17\pi}{20} + i \sin \frac{17\pi}{20} \right). \]

49. \[ \sqrt[3]{2} \left( \cos \frac{2k\pi + \pi/2}{5} + i \sin \frac{2k\pi + \pi/2}{5} \right) = 2 \left( \cos \frac{\pi(4k + 1)}{10} + i \sin \frac{\pi(4k + 1)}{10} \right), \]
\[ k = 0, 1, 2, 3, 4: \]
\[ \sqrt[3]{2} \left( \cos \frac{9\pi}{10} + i \sin \frac{9\pi}{10} \right), \sqrt[3]{2} \left( \cos \frac{13\pi}{10} + i \sin \frac{13\pi}{10} \right). \]
For #51–56, the nth roots of $r(\cos \theta + i \sin \theta)$ are
\[
\sqrt[n]{r(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n})}, \quad k = 0, 1, 2, \ldots, n-1.
\]

50. \[
\sqrt[5]{\left(\cos \frac{2k\pi + \pi/3}{5} + i \sin \frac{2k\pi + \pi/3}{5}\right)} = \sqrt[5]{\left(\cos \frac{\pi(6k+1)}{15} + i \sin \frac{\pi(6k+1)}{15}\right)}.
\]

51. \[
1 + i = \sqrt[4]{\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)}, \quad \text{so the roots are}
\]
\[
\sqrt[4]{\left(\cos \frac{2k\pi + \pi/4}{4} + i \sin \frac{2k\pi + \pi/4}{4}\right)} = \sqrt[4]{\left(\cos \frac{\pi(8k+1)}{16} + i \sin \frac{\pi(8k+1)}{16}\right)}.
\]

52. \[
1 - i = \sqrt[3]{\left(\cos \frac{\pi}{4} + i \sin \frac{7\pi}{4}\right)}, \quad \text{so the roots are}
\]
\[
\sqrt[3]{\left(\cos \frac{2k\pi + 7\pi/4}{3} + i \sin \frac{2k\pi + 7\pi/4}{3}\right)} = \sqrt[3]{\left(\cos \frac{\pi(8k+7)}{24} + i \sin \frac{\pi(8k+7)}{24}\right)}.
\]

53. \[
2 + 2i = 2\sqrt[4]{\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)}, \quad \text{so the roots are}
\]
\[
\sqrt[4]{\left(\cos \frac{2k\pi + \pi/4}{3} + i \sin \frac{2k\pi + \pi/4}{3}\right)} = \sqrt[4]{\left(\cos \frac{\pi(8k+1)}{12} + i \sin \frac{\pi(8k+1)}{12}\right)}, \quad k = 0, 1, 2;
\]
\[
\sqrt[4]{\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)}, \quad -1 + i,
\frac{\pi}{12} + i \sin \frac{17\pi}{12}.
\]

54. \[
-2 + 2i = 2\sqrt[4]{\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)}, \quad \text{so the roots are}
\]
\[
\sqrt[4]{\left(\cos \frac{2k\pi + 3\pi/4}{4} + i \sin \frac{2k\pi + 3\pi/4}{4}\right)} = \sqrt[4]{\left(\cos \frac{\pi(8k+3)}{16} + i \sin \frac{\pi(8k+3)}{16}\right)}, \quad k = 0, 1, 2, 3;
\]
\[
\sqrt[4]{\left(\cos \frac{3\pi}{16} + i \sin \frac{3\pi}{16}\right)}, \quad \sqrt[4]{\left(\cos \frac{11\pi}{16} + i \sin \frac{11\pi}{16}\right)}, \quad \sqrt[4]{\left(\cos \frac{27\pi}{16} + i \sin \frac{27\pi}{16}\right)}.
\]

55. \[
-2i = 2\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right), \quad \text{so the roots are}
\]
\[
\sqrt[5]{\left(\cos \frac{2k\pi + 2\pi}{5} + i \sin \frac{2k\pi + 2\pi}{5}\right)} = \sqrt[5]{\left(\cos \frac{\pi(8k+3)}{10} + i \sin \frac{\pi(8k+3)}{10}\right)}, \quad k = 0, 1, 2, 3, 4,
\frac{1 + i}{\sqrt[4]{2}}, \quad \sqrt[4]{\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right)}, \quad \sqrt[4]{\left(\cos \frac{7\pi}{10} + i \sin \frac{7\pi}{10}\right)}, \quad \sqrt[4]{\left(\cos \frac{5\pi}{10} + i \sin \frac{5\pi}{10}\right)}, \quad \sqrt[4]{\left(\cos \frac{3\pi}{10} + i \sin \frac{3\pi}{10}\right)}.
\]

56. \[
32 = 32(\cos 0 + i \sin 0), \quad \text{so the roots are}
\]
\[
\sqrt[6]{32\left(\cos \frac{2k\pi + 0}{5} + i \sin \frac{2k\pi + 0}{5}\right)} = 2\left(\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}\right), \quad k = 0, 1, 2, 3, 4;
\]
\[
\left(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}\right), \quad \left(\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}\right), \quad \left(\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}\right).
\]

For #57–60, the nth roots of unity are
\[
\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1.
\]

57. \[
\frac{1}{2} = \pm \frac{\sqrt{3}}{2}.
\]
58. $\pm 1, \pm i$

59. $\pm 1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

60. $1, -1$

61. $z = (1 + \sqrt{3}i)^3 = \left[2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)\right]^3$
   $= 8(\cos \pi + i \sin \pi) = -8$; the cube roots are $-2$ and $1 \pm \sqrt{3}i$.

62. $z = (-2 - 2i)^4 = \left[2\sqrt{2} \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)\right]^4$
   $= 64(\cos 5\pi + i \sin 5\pi) = -64$; the fourth roots are $2 \pm 2i$ and $-2 \pm 2i$.

63. \[
\frac{z_1}{z_2} = \frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)} = \frac{r_1 \cos \theta_1 + i \sin \theta_1}{r_2 \cos \theta_2 + i \sin \theta_2} \frac{\cos \theta_2 - i \sin \theta_2}{\cos \theta_2 - i \sin \theta_2}
= \frac{r_1}{r_2} \left[ \cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2) \right].
\]

Now use the angle difference formulas:
$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$ and $\sin(\theta_1 - \theta_2) = \sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2$.

So \[
\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].
\]

64. The $n$th roots are given by $\sqrt[n]{r} \left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}\right)$, $k = 0, 1, 2, \ldots, n - 1$:
$\sqrt[n]{r} \left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right), \sqrt[n]{r} \left(\cos \frac{\theta + 2\pi}{n} + i \sin \frac{\theta + 2\pi}{n}\right)$,
$\sqrt[n]{r} \left(\cos \frac{\theta + 4\pi}{n} + i \sin \frac{\theta + 4\pi}{n}\right), \ldots, \sqrt[n]{r} \left(\cos \frac{\theta + 2(n-1)\pi}{n} + i \sin \frac{\theta + 2(n-1)\pi}{n}\right)$.

The angles between successive values in this list differ by $\frac{2\pi}{n}$ radians, while the first and last roots differ by $2\pi - \frac{2\pi}{n}$, which also makes the angle between them $\frac{2\pi}{n}$. Also, the modulus of each root is $\sqrt[n]{r}$, placing it at that distance from the origin, on the circle with radius $\sqrt[n]{r}$.

65. False. If $z = r(\cos \theta + i \sin \theta)$, then it is also true that $z = r(\cos (\theta + 2n\pi) + i \sin (\theta + 2n\pi))$ for any integer $n$.

For example, here are two trigonometric forms for $1 + i$:
$\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right), \sqrt{2} \left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}\right)$.

66. True. $i^3 = i^2 \cdot i = -i$, so $i$ is a cube root of $-i$.

67. $2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) = 2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = -1 + \sqrt{3}i$

The answer is B.
68. Any complex number has \( n \) distinct \( n \)th roots, so \( 1 - i \) has five 5th roots. The answer is E.

69. \[
\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \cdot \sqrt{2} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right)
\]
= \[
\sqrt{2} \cdot \sqrt{2} \left[ \cos \left( \frac{\pi}{4} + \frac{7\pi}{4} \right) + i \sin \left( \frac{\pi}{4} + \frac{7\pi}{4} \right) \right]
\]
= \[
2 \left( \cos 2\pi + i \sin 2\pi \right)
\]
= \[
2
\]
The answer is A.

70. \((\sqrt{7})^4 = \left( (\sqrt{7})^2 \right)^2 = 7^2 = 1 \neq 1; \text{The answer is E.}\)

71. (a) \( a + bi = r(\cos \theta + i \sin \theta) \), where \( r = \sqrt{a^2 + b^2} \) and \( \theta = \tan^{-1}\left( \frac{b}{a} \right) \). Then \( a + (-bi) = r(\cos \theta' + i \sin \theta') \), where \( r' = \sqrt{a^2 + b^2} \) and \( \theta' = \tan^{-1}\left( \frac{-b}{a} \right) \). Since \( r' = \sqrt{a^2 + b^2} = r \) and \( \theta' = \tan^{-1}\left( \frac{-b}{a} \right) = -\tan^{-1}\left( \frac{b}{a} \right) = -\theta \), we have \( a - bi = r(\cos (-\theta) + i \sin (-\theta)) \).

(b) \( z \cdot \bar{z} = r(\cos \theta + i \sin \theta) \cdot r(\cos (-\theta) + i \sin (-\theta)) \)
= \[
r^2 \left[ \cos \theta \cos (-\theta) + i \sin \theta \sin (-\theta) \right] = r^2 \sin^2 \theta
\]
Recall, however, that \((a, b)\) is in the quadrant directly opposite the quadrant that holds \((a, b)\) (i.e., if \((a, b)\) is in Quadrant I, \((-a, -b)\) is in Quadrant III, and if \((a, b)\) is in Quadrant II, \((-a, -b)\) is in Quadrant IV). Thus, \(-z = \sqrt{a^2 + b^2} (\cos(\theta + \pi) + i \sin (\theta + \pi)) = r(\cos(\theta + \pi) + i \sin(\theta + \pi)) \).

72. (a) \[
|z| = \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2}
\]
= \[
\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta}
\]
= \[
r \sqrt{\cos^2 \theta + \sin^2 \theta}
\]
= \[
|z|
\]

(b) \[
|z_1 \cdot z_2| = |[r_1 \cos \theta_1 + (r_1 \sin \theta_1)i] \cdot [r_2 \cos \theta_2 + (r_2 \sin \theta_2)i]| = |r_1 r_2 \cos \theta_1 \cos \theta_2 + (r_1 r_2 \sin \theta_1 \sin \theta_2)i + (r_1 r_2 \sin \theta_1 \cos \theta_2)i + (r_1 r_2 \sin \theta_1 \cos \theta_2)i|
\]
= \[
r_1 r_2 [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)i]
\]
= \[
r_1 r_2 [\cos (\theta_1 + \theta_2) + (\sin (\theta_1 + \theta_2))i]
\]
= \[
\sqrt{(r_1 r_2)^2 [\cos^2 (\theta_1 + \theta_2) + \sin^2 (\theta_1 + \theta_2)]}
\]
= \[
\sqrt{(r_1 r_2)^2}
\]
= \[
|r_1| \cdot |r_2|
\]
= \[
|z_1| \cdot |z_2|
\]
by part (a)

73. (a) \[
25f(2)^* \{ \cos(\pi/4) \} \ast \{ \sin(\pi/4) \} \ast 14 \ast \{ \cos(\pi/3) \ast \sin(\pi/3) \}
\]
= 478.11 + 128.11i

(b) \[
2f(2)^* \{ \cos(135^\circ) \ast \sin(30^\circ) \} \ast \langle 6^\circ \rangle \{ \cos(300^\circ) \ast \sin(300^\circ) \}
\]
= -46.46 - 12.12i

(c) \[
\cos(3\pi/4) + \i \cdot \sin(3\pi/4) \ast 2.00 \ast 1.3i
\]
74. (a) \[ x(t) = \cos(2\pi t), \quad y(t) = \sin(2\pi t), \quad \text{for} \quad t \in [-2.4, 2.4] \text{ by } [-1.6, 1.6] \]

(b) Yes. \(6\pi/8, 10\pi/8, 14\pi/8\)

(c) For fifth and seventh roots of unity, all of the roots except the complex number 1 generate the corresponding roots of unity. For the sixth roots of unity, only \(2\pi/6\) and \(10\pi/6\) generate the sixth roots of unity.

(d) \(2\pi k / n\) generates the \(n\)th roots of unity if and only if \(k\) and \(n\) have no common factors other than 1.

75. Using \(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.62\), we have \(\sqrt{2} + i \approx \sqrt{3} (\cos 0.62 + i \sin 0.62)\), so graph \(x(t) = (\sqrt{3})^t \cos(0.62t)\) and \(y(t) = (\sqrt{3})^t \sin(0.62t)\).

Use \(T_{\text{min}} = 0, T_{\text{max}} = 4, T_{\text{step}} = 1\). Shown is \([-7, 2]\) by \([-0, 6]\).

76. \(-1 + i = \sqrt{2} (\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})\), so graph \(x(t) = (\sqrt{2})^t \cos(0.75\pi t)\) and \(y(t) = (\sqrt{2})^t \sin(0.75\pi t)\).

Use \(T_{\text{min}} = 0, T_{\text{max}} = 4, T_{\text{step}} = 1\). Shown is \([-5, 3]\) by \([-3, 3]\).

77. Suppose that \(z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)\) and \(z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)\). Each triangle’s angle at the origin has the same measure: For the smaller triangle, this angle has measure \(\theta_1\); for the larger triangle, the side from the origin out to \(z_2\) makes an angle of \(\theta_2\) with the \(x\)-axis, while the side from the origin to \(z_1z_2\) makes an angle of \(\theta_1 + \theta_2\), so that the angle between is \(\theta_1\) as well.

The corresponding side lengths for the sides adjacent to these angles have the same ratio: two longest side have lengths \(|z_1| = r_1\) (for the smaller) and \(|z_1z_2| = r_1r_2\), for a ratio of \(r_2\). For the other two sides, the lengths are 1 and \(r_2\), again giving a ratio of \(r_2\). Finally, the Law of Sines can be used to show that the remaining side have the same ratio.

78. Construct an angle with vertex at \(z_2\), and one ray from \(z_2\) to 0, congruent to the angle formed by 0, 1, and \(z_1\), with vertex 1. Also, be sure that this new angle is oriented in the appropriate direction: e.g., if \(z_2\) is located “counterclockwise” from 1 then the points on this new ray should also be located counterclockwise from \(z_2\). Now similarly construct an angle with vertex at 0, and one ray from 0 to \(z_2\), congruent to the angle formed by 1, 0, and \(z_1\), with vertex 0. The intersection of the two newly constructed rays is \(z_1z_2\).

79. The solutions are the cube roots of 1:
\[
\cos\left(\frac{2\pi k}{3}\right) + i \sin\left(\frac{2\pi k}{3}\right), \quad k = 0, 1, 2 \text{ or } 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i.
\]

80. The solutions are the fourth roots of 1:
\[
\cos\left(\frac{\pi k}{2}\right) + i \sin\left(\frac{\pi k}{2}\right), \quad k = 0, 1, 2, 3 \text{ or } -1, 1, -i, i.
\]

81. The solutions are the cube roots of \(-1\):
\[
\cos\left(\frac{\pi + 2\pi k}{3}\right) + i \sin\left(\frac{\pi + 2\pi k}{3}\right), \quad k = 0, 1, 2 \text{ or } -1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i.
\]

82. The solutions are the fourth roots of \(-1\):
\[
\cos\left(\frac{\pi + 2\pi k}{4}\right) + i \sin\left(\frac{\pi + 2\pi k}{4}\right), \quad k = 0, 1, 2, 3 \text{ or } \frac{\sqrt{2} + \sqrt{2}}{2}, \frac{\sqrt{2} - \sqrt{2}}{2}, -\frac{\sqrt{2} + \sqrt{2}}{2}, -\frac{\sqrt{2} - \sqrt{2}}{2}.
\]

83. The solutions are the fifth roots of \(-1\):
\[
\cos\left(\frac{\pi + 2\pi k}{5}\right) + i \sin\left(\frac{\pi + 2\pi k}{5}\right), \quad k = 0, 1, 2, 3, 4 \text{ or } -1, \approx 0.81 + 0.59i, 0.81 - 0.59i, -0.31 + 0.95i, -0.31 - 0.95i.
\]

84. The solutions are the fifth roots of 1:
\[
\cos\left(\frac{2\pi k}{5}\right) + i \sin\left(\frac{2\pi k}{5}\right), \quad k = 0, 1, 2, 3, 4 \text{ or } 1, \approx 0.31 + 0.95i, 0.31 - 0.95i, -0.81 + 0.59i, -0.81 - 0.59i.
\]

### Chapter 6 Review

1. \(u - v = (2 - 4, -1 - 2) = (-2, -3)\)
2. \(2u - 3w = (4 - 3, -2 + 9) = (1, 7)\)
3. \(|u + v| = \sqrt{(2 + 4)^2 + (-1 + 2)^2} = \sqrt{37}\)
4. \(|w - 2u| = \sqrt{(1 - 4)^2 + (-3 + 2)^2} = \sqrt{10}\)
5. \( \mathbf{u} \cdot \mathbf{v} = 8 - 2 = 6 \)
6. \( \mathbf{u} \cdot \mathbf{w} = 2 + 3 = 5 \)
7. \( 3\mathbf{AB} = 3(3 - 2, 1 - (-1)) = (3, 6) \);
   \[ |3\mathbf{AB}| = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5} \]
8. \( \mathbf{AB} + \mathbf{CD} = (3 - 2, 1 - (-1)) + (1 - (-4), -5 - 2) = (6, -5) \); \( |\mathbf{AB} + \mathbf{CD}| = \sqrt{6^2 + 5^2} = \sqrt{61} \)
9. \( \mathbf{AC} + \mathbf{BD} = (-4 - 2, 2 - (-1)) + (1 - 3, -5 - 1) = (-8, -3) \); \( |\mathbf{AC} + \mathbf{BD}| = \sqrt{8^2 + 3^2} = \sqrt{73} \)
10. \( \mathbf{CD} - \mathbf{AB} = (1 - (-4), -5 - 2) - (3 - 2, 1 - (-1)) = (4, -9) \); \( |\mathbf{CD} - \mathbf{AB}| = \sqrt{4^2 + 9^2} = \sqrt{97} \)

11. (a) \( \frac{\mathbf{AB}}{|\mathbf{AB}|} = \frac{(-2, 1)}{\sqrt{(-2)^2 + 1^2}} = \frac{(-2, 1)}{\sqrt{5}} = \left( -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \)
    (b) \( -3 \cdot \frac{\mathbf{AB}}{|\mathbf{AB}|} = -3 \cdot \frac{(-2, 1)}{\sqrt{5}} = \left( -\frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}} \right) \)
12. (a) \( \frac{\mathbf{BA}}{|\mathbf{BA}|} = \frac{(2, 0)}{\sqrt{2^2 + 0^2}} = \left( \frac{2}{2}, 0 \right) = (1, 0) \)
    (b) \( -3 \cdot \frac{\mathbf{BA}}{|\mathbf{BA}|} = -3 \cdot (1, 0) = (-3, 0) \)

For #13–14, the direction angle \( \theta \) of \( (a, b) \) has tan \( \theta = \frac{b}{a} \), start with tan \( ^{-1} \left( \frac{b}{a} \right) \), and add (or subtract) 180° if the angle is not in the correct quadrant. The angle between two vectors is the absolute value of the difference between their angles; if this difference is greater than 180°, subtract it from 360°.

13. (a) \( \theta_u = \tan^{-1} \left( \frac{3}{4} \right) \approx 0.64 \), \( \theta_v = \tan^{-1} \left( \frac{5}{2} \right) \approx 1.19 \)
    (b) \( \theta_v - \theta_u \approx 0.55 \).
14. (a) \( \theta_u = \pi + \tan^{-1}(-2) = -\cos^{-1} \left( -\frac{1}{\sqrt{5}} \right) \approx 2.03 \),
    \( \theta_v = \tan^{-1} \left( \frac{2}{3} \right) \approx 0.59 \)
    (b) \( \theta_v - \theta_u \approx 1.45 \).
15. \( (-2.5 \cos 25°, -2.5 \sin 25°) \approx (-2.27, -1.06) \)
16. \( (-3.1 \cos 135°, -3.1 \sin 135°) = (1.55\sqrt{2}, -1.55\sqrt{2}) \)
17. \( 2 \cos(-\pi/4), 2 \sin(-\pi/4) = (\sqrt{2}, -\sqrt{2}) \)
18. \( 3.6 \cos(3\pi/4), 3.6 \sin(3\pi/4) = (-1.8\sqrt{2}, 1.8\sqrt{2}) \)
19. \( (1, -\frac{2\pi}{3} + (2n + 1)\pi) \) and \( (1, -\frac{2\pi}{3} + 2n\pi) \) \( n \) an integer
20. \( (2, \frac{5\pi}{6} + (2n + 1)\pi) \) and \( (2, \frac{5\pi}{6} + 2n\pi) \) \( n \) an integer
21. (a) \( \sqrt{3}, \pi + \tan^{-1} \left( -\frac{3}{2} \right) \approx (\sqrt{3}, 2.16) \) or \( \sqrt{3}, 2\pi + \tan^{-1} \left( -\frac{3}{2} \right) \approx (\sqrt{3}, 5.30) \)
    (b) \( \sqrt{3}, \tan^{-1} \left( -\frac{3}{2} \right) \approx (\sqrt{3}, -0.98) \) or \( -\sqrt{3}, \pi + \tan^{-1} \left( -\frac{3}{2} \right) \approx (-\sqrt{3}, 2.16) \)

(c) The answers from (a), and also
   \( -\sqrt{3}, 3\pi + \tan^{-1} \left( -\frac{3}{2} \right) \approx (\sqrt{3}, 8.44) \) or
   \( \sqrt{3}, 4\pi + \tan^{-1} \left( -\frac{3}{2} \right) \approx (\sqrt{3}, 11.58) \)
22. (a) \( (-10, 0) \) or \( (10, \pi) \) or \( (-10, 2\pi) \)
    (b) \( (10, -\pi) \) or \( (-10, 0) \) or \( (10, \pi) \)
    (c) The answers from (a), and also \( (10, 3\pi) \) or \( (-10, 4\pi) \)
23. (a) \( (5, 0) \) or \( (-5, \pi) \) or \( (5, 2\pi) \)
    (b) \( (-5, -\pi) \) or \( (5, 0) \) or \( (-5, \pi) \)
    (c) The answers from (a), and also \( (-5, 3\pi) \) or \( (5, 4\pi) \)
24. (a) \( (-2, \frac{\pi}{2}) \) or \( (2, \frac{3\pi}{2}) \)
    (b) \( 2, -\frac{\pi}{2} \) or \( -2, -\frac{\pi}{2} \)
    (c) The answers from (a), and also \( -2, \frac{5\pi}{2} \)
25. \( \tan^{-1} \sqrt{x^2 + y^2} \approx \frac{3}{5} \), so \( y = 4 + 3 \left( -\frac{1}{5} \right) \)
    \( = -\frac{3}{5} x + \frac{29}{5} \)

Line through \( (0, 29/5) \) with slope \( m = -\frac{3}{5} \)
26. \( t = x - 4, y = -8 - 5(x - 4) = -5x + 12, 1 \leq x \leq 9 \); segment from \( (1, 7) \) to \( (9, -33) \)
27. \( t = y + 1, x = 2(y + 1)^2 + 3 \); Parabola that opens to right with vertex at \( (3, -1) \).
28. \( x^2 + y^2 = (3 \cos t)^2 + (3 \sin t)^2 = 9 \cos^2 t + 9 \sin^2 t = 9 \), so \( x^2 + y^2 = 9 \); Circle of radius 3 centered at \( (0, 0) \).
29. \( x + 1 = e^{5t}, t = \frac{\ln(x + 1)}{2} \), so \( y = e^{\ln(x^2 + 1/3)} \)
   \( = \sqrt{x + 1} \); square root function starting at \( (1, 0) \)
30. \( t = \frac{\ln(x + 1/3)}{2} \)
31. \( m = \frac{4 - (-2)}{4 - (-1)} = 3 \), so \( \Delta x = 2 \) when \( \Delta y = 3 \).
   One possibility for the parametrization of the line is: \( x = 2t + 3, y = 3t + 4 \).
32. \( m = \frac{1 - 3}{5 - (-2)} = -\frac{2}{7} \), so \( \Delta x = 7 \) when \( \Delta y = -2 \).
   One possibility for the parametrization of the segment is: \( x = 7t + 5, y = -2t + 1, -1 \leq t \leq 0 \) Another possibility is \( x = 7t - 2, y = -2t + 3, 0 \leq t \leq 1 \).
33. \( a = -3, b = 4, |z| = \sqrt{3^2 + 4^2} = 5 \)
34. \( z_1 = 5 \left\{ \cos \left[ \cos^{-1} \left( -\frac{3}{5} \right) \right] + i \sin \left[ \cos^{-1} \left( -\frac{3}{5} \right) \right] \right\} \)
   \( \approx 5 [\cos(2.21) + i \sin(2.21)] \)
35. \( 6 (\cos 30° + i \sin 30°) = 6 \left( \frac{\sqrt{3}}{2} + \frac{1}{2} i \right) = 3\sqrt{3} + 3i \)
36. \( 3 (\cos 150° + i \sin 150°) = 3 \left( -\frac{\sqrt{3}}{2} + \frac{1}{2} i \right) = -1.5\sqrt{3} + 1.5i \)
37. \(2.5 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) = 2.5 \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) = -1.25 - 1.25\sqrt{3} i\)

38. \(4 \left( \cos 2.5 + i \sin 2.5 \right) \approx -3.2 + 2.39i\)

39. \(3 - 3i = 3\sqrt{2} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right)\). Other representations would use angles \(\frac{7\pi}{4} + 2n\pi\), \(n\) an integer.

40. \(-1 + i\sqrt{2} = \sqrt{3} \left\{ \cos \left( -\frac{1}{\sqrt{3}} \right) + i \sin \left( -\frac{1}{\sqrt{3}} \right) \right\} = \sqrt{3} \cos (2.19) + i (2.19)\). Other representations would use angles \(2.19 + 2n\pi\), \(n\) an integer.

41. \(5 - 3i = \sqrt{34} \left\{ \cos \left[ \tan^{-1} \left( -\frac{5}{3} \right) \right] + i \sin \left[ \tan^{-1} \left( -\frac{5}{3} \right) \right] \right\} = \sqrt{34} \cos (-1.03) + i \sin (-1.03) \approx \sqrt{34} \cos (5.25 + i \sin (5.25)) \approx \sqrt{34} \cos (5.25)\). Other representations would use angles \(5.25 + 2n\pi\), \(n\) an integer.

42. \(-2 - 2i = 2\sqrt{2} \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right)\). Other representations would use angles \(\frac{5\pi}{4} + 2n\pi\), \(n\) an integer.

43. \(z_1z_2 = (3)(4) \left( \cos (30^o + 60^o) + i \sin (30^o + 60^o) \right) = 12 \left( \cos 90^o + i \sin 90^o \right)\)

\[\begin{align*}
z_1/z_2 &= \frac{3}{4} \left[ \cos (30^o - 60^o) + i \sin (30^o - 60^o) \right] = \frac{3}{4} \left[ \cos (-30^o) + i \sin (-30^o) \right] = \frac{3}{4} \left( \cos 330^o + i \sin 330^o \right) = \frac{3}{4} \left( \frac{1}{2} + i \frac{-\sqrt{3}}{2} \right) = \frac{3}{8} + \frac{3\sqrt{3}}{8} i.
\end{align*}\]

44. \(z_1z_2 = (5)(-2) \left( \cos (20^o + 45^o) + i \sin (20^o + 45^o) \right) = -10 \left( \cos 65^o + i \sin 65^o \right) = 10 \left( \cos 245^o + i \sin 245^o \right)\)

\[\begin{align*}
z_1/z_2 &= \frac{5}{2} \left[ \cos (20^o - 45^o) + i \sin (20^o - 45^o) \right] = -2.5 \left( \cos (-25^o) + i \sin (-25^o) \right) = 2.5 \left( \cos 155^o + i \sin 155^o \right) = 2.5 \left( \frac{\sqrt{2}}{2} + i \frac{-\sqrt{2}}{2} \right) = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2} i.
\end{align*}\]

45. \(a) \left[ 3 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^5 = 3^5 \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = 243 \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = 243 \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = \frac{-243\sqrt{2}}{2} + \frac{-243\sqrt{2}}{2} i.
\]

\(b) \left[ 2 \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right) \right]^8 = 2^8 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) = 256 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) = -128 + 128\sqrt{3} i.
\]

46. \(a) \left[ 5 \left( \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) \right]^3 = 5^3 \left( \cos 5\pi + i \sin 5\pi \right) = 125 \left( \cos \pi + i \sin \pi \right) = -125 + 0i = -125.
\]

\(b) \left[ 5 \left( \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) \right]^3 = 5^3 \left( \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) = 125 \left( \cos \pi + i \sin \pi \right) = -125 + 0i = -125.
\]
51. \(1 = \cos 0 + i \sin 0\), so the roots are 
\[
\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5},
\]
\(k = 0, 1, 2, 3, 4, 5; \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}, \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}, \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}
\]

52. \(-1 = \cos \pi + i \sin \pi\), so the roots are 
\[
\cos \frac{2k\pi}{6} + i \sin \frac{2k\pi}{6} = \cos \frac{2k\pi}{6} + i \sin \frac{2k\pi}{6},
\]
\(k = 0, 1, 2, 3, 4, 5; \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}, \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}, \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}, \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}, \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i, \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}
\]

53. Graph (b)
54. Not shown
55. Graph (a)
56. Not shown
57. Not shown
58. Graph (d)
59. Graph (c)
60. Not shown
61. \(x^2 + y^2 = 4\) — a circle with center \((0, 0)\) and radius 2

62. \(r^2 + 2r \sin \theta = x^2 + y^2 + 2y = 0\). Completing the square: 
\[
x^2 + (y^2 + 1)^2 = 1 — a circle of radius 1 with center \((0, -1)\)
\]

63. \(r^2 + 3r \cos \theta + 2r \sin \theta = x^2 + y^2 + 3x + 2y = 0\). Completing the square: 
\[
\left(x + \frac{3}{2}\right)^2 + (y + 1)^2 = \frac{13}{4}
\]
—a circle of radius \(\frac{\sqrt{13}}{2}\) with center \((-\frac{3}{2}, -1)\)

64. \(1 - \frac{3}{r \cos \theta} = 1 - \frac{3}{x} = 0 \Rightarrow x - 3 = 0, x = 3\) — a vertical line through \((3, 0)\)

65. \(r = -\frac{4}{\sin \theta} = -4 \csc \theta\)

66. \(r = \frac{5}{\cos \theta} = 5 \sec \theta\)

67. \((r \cos \theta - 3)^2 + (r \sin \theta + 1)^2 = 10\), so 
\[
r^2 (\cos^2 \theta + \sin^2 \theta) + r (-6 \cos \theta + 2 \sin \theta) + 10 = 10, \text{ or } r = 6 \cos \theta - 2 \sin \theta
\]

68. \(2r \cos \theta - 3r \sin \theta = 4, r = \frac{4}{2 \cos \theta - 3 \sin \theta}\)
69. Domain: All reals
   Range: \([-3, 7]\)
   Symmetric about the y-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 7
   No asymptotes

70. Domain: All reals
   Range: \([0, 8]\)
   Symmetric about the x-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 8
   No asymptotes

71. Domain: All reals
   Range: \([-2, 2]\)
   Symmetric about the y-axis
   Continuous
   Bounded
   Maximum \(r\)-value: 2
   No asymptotes

72. Domain: \([\frac{\pi}{2}, \pi] \cup \left[\frac{3\pi}{2}, 2\pi\right]\)
   Range: \([0, \sqrt{2}]\)
   Symmetric about the origin
   Bounded
   Maximum \(r\)-value: \(\sqrt{2}\)
   No asymptotes

73. (a) \(r = a \sec \theta \Rightarrow \frac{r}{\sec \theta} = a \Rightarrow r \cos \theta = a \Rightarrow x = a.\)
   (b) \(r = b \csc \theta \Rightarrow \frac{r}{\csc \theta} = b \Rightarrow r \sin \theta = b \Rightarrow y = b.\)
   (c) \(y = mx + b \Rightarrow r \sin \theta = mr \cos \theta + b \Rightarrow \)
   \(r(\sin \theta - m \cos \theta) = b \Rightarrow r = \frac{b}{\sin \theta - m \cos \theta}.\)
   The domain of \(r\) is any value of \(\theta\) for which
   \(\sin \theta \neq m \cos \theta \Rightarrow \tan \theta \neq m \arctan \left(\frac{m}{\cot \theta} \right).\)
   (d)

74. (a) \(\mathbf{v} = 540 \begin{pmatrix} \sin 80^\circ \\ \cos 80^\circ \end{pmatrix} \approx (531.80, 93.77)\)
   (b) The wind vector is \(\mathbf{w} = 55 \begin{pmatrix} \sin 100^\circ \\ \cos 100^\circ \end{pmatrix} \approx (54.16, -9.55)\).
   Actual speed: \(\|\mathbf{v} + \mathbf{w}\| \approx \sqrt{585.96^2 + 84.22^2} \approx 591.98\) mph.
   Actual bearing: \(\tan^{-1} \left(\frac{585.96}{84.22} \right) \approx 82.82^\circ\).

75. (a) \(\mathbf{v} = 480 \begin{pmatrix} \sin 285^\circ \\ \cos 285^\circ \end{pmatrix} \approx (-463.64, 124.23)\)
   (b) The wind vector is \(\mathbf{w} = 30 \begin{pmatrix} \sin 265^\circ \\ \cos 265^\circ \end{pmatrix} \approx (-29.89, -2.61)\).
   Actual speed: \(\|\mathbf{v} + \mathbf{w}\| \approx \sqrt{493.53^2 + 121.62^2} \approx 508.29\) mph.
   Actual bearing: \(360^\circ + \tan^{-1} \left(\frac{-493.53}{121.62} \right) \approx 283.84^\circ\).

76. \(\mathbf{F} = (120 \cos 20^\circ, 120 \sin 20^\circ) + (300 \cos(-5^\circ), 300 \sin(-5^\circ)) \approx (411.62, 14.90)\), so
   \(\|\mathbf{F}\| \approx \sqrt{411.62^2 + 14.90^2} \approx 411.89\) lb and
   \(\theta = \tan^{-1} \left(\frac{14.90}{411.62} \right) \approx 2.07^\circ\).

77. \(\mathbf{F}_1, \mathbf{F}_2\) Force to keep car from going downhill
   (a) \(\mathbf{F}_1 = -3000 \cos 16^\circ \approx -826.91\), so the force required to keep the car from rolling down the hill is approximately 826.91 pounds.
   (b) \(\mathbf{F}_2 = -3000 \cos 16^\circ \approx 2883.79\), so the force perpendicular to the street is approximately 2883.79 pounds.

78. \(\mathbf{F} = 36 \cdot \frac{(3, 5)}{\sqrt{3^2 + 5^2}} = (108, 180) \cdot \frac{108}{\sqrt{34}}\)
   Since \(\overrightarrow{AB} = (10, 0), \mathbf{F} \cdot \overrightarrow{AB} = (108 \cdot \frac{108}{\sqrt{34}}) + 0\)
   \(= \frac{1080}{\sqrt{34}} \approx 185.22\) foot-pounds.
79. (a) \( h = -16t^2 + v_0t + s_0 = -16t^2 + 245t + 200 \)

(b) Graph and trace: \( x = 17 \) and 
\( y = -16t^2 + 245t + 200 \) with \( 0 \leq t \leq 16.1 \) (upper limit may vary) on \([0, 18]\) by \([0, 1200]\). This graph will appear as a vertical line from about \((17, 0)\) to about \((17, 1138)\). Tracing shows how the arrow begins at a height of 200 ft, rises to over 1000 ft, then falls back to the ground.

(c) Graph \( x = t \) and \( y = -16t^2 + 245t + 200 \) with 
\( 0 \leq t \leq 16.1 \) (upper limit may vary).

(d) When \( t = 4 \), \( h = 924 \) ft.

(e) When \( t \approx 7.66 \), the arrow is at its peak: about 1138 ft.

(f) The arrow hits the ground \((h = 0)\) after about 16.09 sec.

80. \( x = 35 \cos \left( \frac{\pi}{10}t \right), \ y = 50 + 35 \sin \left( \frac{\pi}{10}t \right), \) assuming the wheel turns counterclockwise.

81. \( x = 40 \sin \left( \frac{2\pi}{15}t \right), \ y = 50 - 40 \cos \left( \frac{2\pi}{15}t \right), \) assuming the wheel turns counterclockwise.

82. \( x = -40 \sin \left( \frac{\pi}{9}t \right), \ y = 50 + 40 \cos \left( \frac{\pi}{9}t \right), \) assuming the wheel turns counterclockwise.

83. (a) 

(b) All 4's should be changed to 5's.

84. \( x = (66 \cos 5^\circ)y \) and \( y = -16t^2 + (66 \sin 5^\circ)t + 4, \)
\( y = 0 \) when \( t \approx 0.71 \) sec (and also when \( t \approx -0.352, \) but that is not appropriate in this problem). When 
\( t \approx 0.71 \) sec, \( x \approx 46.75 \) ft.

85. \( x = (66 \cos 12^\circ)t \) and \( y = -16t^2 + (66 \sin 12^\circ)t + 3.5, \)
\( y = 0 \) when \( t \approx 1.06 \) sec (and also when \( t \approx -0.206, \) but that is not appropriate in this problem). When 
\( t \approx 1.06 \) sec, \( x \approx 68.65 \) ft.

86. \( x = (70 \cos 45^\circ)t \) and \( y = -16t^2 + (70 \sin 45^\circ)t. \)
The ball traveled 40 yd (120 ft) horizontally after about 2.42 sec, at which point it is about 25.96 ft above the ground, so it clears the crossbar.

87. If we assume that the initial height is 0 ft, then 
\( x = (85 \cos 56^\circ)t \) and \( y = -16t^2 + (85 \sin 56^\circ)t. \)
[If the assumed initial height is something other than 0 ft, add that amount to \( y \).

(a) Find graphically: the maximum \( y \) value is about 77.59 ft (after about 2.20 seconds).

(b) \( y = 0 \) when \( t \approx 4.404 \) sec

88. \( x = (v_0 \cos 30^\circ)t \) and \( y = -16t^2 + (v_0 \sin 30^\circ)t + 2.5. \)
\( v_0 \) must be (at least) just over 125 ft/sec. This can be found graphically (by trial-and-error), or algebraically: the ball is 400 ft from the plate \((i.e., x = 400)\) when
\( t = \frac{400}{v_0 \cos 30^\circ} = \frac{800/\sqrt{3}}{v_0} \)
Substitute this value of \( t \) in the parametric equation for \( y \). Then solve to see what value of \( v_0 \) makes \( y \) equal to 15 ft.

The negative root doesn’t apply to this problem, so the initial velocity needed is just over 125 ft/sec.

89. Kathy’s position: \( x_1 = 60 \cos \left( \frac{\pi}{6}t \right) \) and
\( y_1 = 60 + 60 \sin \left( \frac{\pi}{6}t \right) \)
Ball’s position:
\( x_2 = -80 + (100 \cos 70^\circ)t \) and
\( y_2 = -16t^2 + (100 \sin 70^\circ)t \)
Find (graphically) the minimum of 
\( d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). It occurs when
\( t \approx 2.64 \) sec; the minimum distance is about 17.65 ft.

90. \( x = (20 \cos 50^\circ)t \) and \( y = -16t^2 + (20 \sin 50^\circ)t + 5. \)
\( y = 0 \) when \( t = 1.215 \) sec (and also when \( t = -0.257, \) but that is not appropriate in this problem). When
\( t = 1.215 \) sec, \( x = 15.62 \) ft. The dart falls several feet short of the target.

Chapter 6 Project
Answers are based on the sample data shown in the table.

1.

2. Sinusoidal regression produces
\( y \approx 0.28 \sin(3.46x + 1.20) + 0.75 \) or, with a phase shift of \( 2\pi, y \approx 0.28 \sin(3.46x - 5.09) + 0.75 \)
\( \approx 0.28 \sin(3.46(x - 1.47)) + 0.75. \)
3. The curve closely fits the data.

4. The distance and velocity both vary sinusoidally, with the same period but a phase shift of 90° — like the x- and y-coordinates of a point moving around a circle. A scatter plot of distance versus time should have the shape of a circle (or ellipse).